11.4 SOLUTIONS.

6. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$: compare with $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$, which diverges (Harmonic Series)

$$\Rightarrow \quad \frac{1}{n-\sqrt{n}} \overset{?}{>} \frac{1}{n}$$

$$n \overset{?}{>} n-\sqrt{n}$$

$$\sqrt{n} \overset{?}{>} 0 \Rightarrow \text{true for all } n, \text{ thus true for } n \geq 2$$

$\Rightarrow$ Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ diverges and $\dfrac{1}{n-\sqrt{n}} > \dfrac{1}{n}$ for all n, then $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$ diverges.

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n(n+1)(n+2))^{1/3}} = \sum_{n=1}^{\infty} b_n$: use Limit Comparison Test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} a_n$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \left(\frac{\left(\frac{1}{n}\right)}{\left(n(n+1)(n+2)\right)^{1/3}}\right) = \lim_{n\to\infty} \left(\frac{(n^3+3n^2+2n)^{1/3}}{n}\right) =$$

$$\lim_{n\to\infty} \left(\left(\frac{n^3+3n^2+2n}{n^3}\right)^{1/3}\right) = \lim_{n\to\infty} \left(\left(1+\frac{3}{n}+\frac{2}{n^2}\right)^{1/3}\right) = 1$$

$\Rightarrow$ We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and since $\displaystyle\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = 1 > 0$, then Limit Comparison Test tells us $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{(n(n+1)(n+2))^{1/3}}$ diverges as well.

20. $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n} = \sum_{n=1}^{\infty} b_n$: use Limit Comparison Test with $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty} a_n$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \left(\frac{2^n}{3^n} \times \frac{(1+3^n)}{(1+2^n)}\right) = \lim_{n\to\infty} \left(\frac{2^n \times (1+3^n)}{3^n \cdot (1+2^n)}\right) = \lim_{n\to\infty} \left(\frac{\cancel{2^n} \times \cancel{3^n} \left(\frac{1}{3^n}+1\right)}{\cancel{3^n} \times \cancel{2^n} \left(\frac{1}{2^n}+1\right)}\right) =$$

$$= \lim_{n\to\infty} \left(\frac{\frac{1}{3^n}+1}{\frac{1}{2^n}+1}\right) = 1$$

$\Rightarrow$ Since $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ converges (geometric series with $a = 2/3$ and $r = 2/3 < 1$), and $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1$, then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$ converges as well.

26. Optional.

$\sum_{n=1}^{\infty} \frac{n+5}{(n^7+n^2)^{1/3}} = \sum_{n=1}^{\infty} b_n$: use Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}} = \sum_{n=1}^{\infty} a_n$

$$\lim_{n\to\infty}\left(\frac{b_n}{a_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{n+5}{(n^7+n^2)^{1/3}}}{\frac{1}{n^{4/3}}}\right) = \lim_{n\to\infty}\left(\frac{(n+5)}{(n^7+n^2)^{1/3}} \times n^{4/3}\right) =$$

$$\lim_{n\to\infty}\left(\frac{n^{7/3}+5n^{4/3}}{(n^7+n^2)^{1/3}}\right) = \lim_{n\to\infty}\left(\frac{1}{\left(\frac{n^7+n^2}{n^7}\right)^{1/3}} + \frac{5}{\left(\frac{n^7+n^2}{n^4}\right)^{1/3}}\right) =$$

$$\lim_{n\to\infty}\left(\frac{1}{\left(1+\frac{1}{n^5}\right)^{1/3}} + \frac{5}{\left(n^3+\frac{1}{n^2}\right)^{1/3}}\right) = 1 + 0 = 1$$

$\Rightarrow$ Since $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ is convergent (p-series with $p = 4/3 > 1$) and since $\lim_{n\to\infty}\left(\frac{b_n}{a_n}\right) = 1 > 0$

then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n+5}{(n^7+n^2)^{1/3}}$ converges as well.

31. Optional.

$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} a_n$: use Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} b_n$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\sin\left(\frac{1}{n}\right)}{1/n}\right) = 1$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1 > 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

32. $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}} = \sum_{n=1}^{\infty} \frac{1}{n \times n^{1/n}} \Rightarrow$ use Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n \times \sqrt[n]{n}} = \sum_{n=1}^{\infty} b_n$, and $\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{n}}{\frac{1}{n \times n^{1/n}}}\right) = \lim_{n\to\infty}\left(\frac{\cancel{n} \times \sqrt[n]{n}}{\cancel{n}}\right) = \lim_{n\to\infty}\left(n^{1/n}\right) = 1 \therefore \sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}} \text{ diverge}$$

$\Rightarrow$ You can show it using L'Hôpital's rule: let $L = \ln(n^{1/n}) = \frac{1}{n}\ln(n)$

$$\therefore \lim_{n\to\infty}\left(\frac{\ln(n)}{n}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{n}}{1}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

$\Rightarrow$ if $\lim_{n\to\infty}(L) \to 0$ and $L = \ln(n^{1/n})$, then $\lim_{n\to\infty}(n^{1/n}) \to 1 \therefore (\ln 1 = 0)$

37. Optional.

$N = 0.d_1 d_2 d_3 d_4 \ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \ldots$, for $d_n \in [0,9] \Rightarrow \sum_{n=1}^{\infty} \frac{d_n}{10^n}$

$\Rightarrow$ consider $\sum_{n=1}^{\infty} \frac{9.5}{10^n} = \frac{9.5}{10} + \frac{9.5}{10^2} + \frac{9.5}{10^3} + \ldots = 0.95\left(1 + \frac{1}{10} + \frac{1}{10^2} + \ldots\right)$

Thus $\sum_{n=1}^{\infty} \frac{9.5}{10^n}$ is a convergent geometric series with $a = 0.95$ and $r = 0.1 < 1$

Claim: $\sum^{n} \frac{9.5}{10^n} = \sum^{n} a_n$ and $a_n > \frac{d_n}{10^n}$

$$\Rightarrow \frac{9.5}{10^n} \overset{?}{>} \frac{d_n}{10^n}$$

$9.5 \overset{?}{>} d_n \Rightarrow$ true, since $d_n \in [0,9]$ and thus: $9 \geq d_n$.

We showed that $a_n > \frac{d_n}{10^n}$ and since $\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=1}^{\infty} \frac{d_n}{10^n}$ converges as well.

38. $\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)} = \sum_{n=2}^{\infty} a_n$: compare with a p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, for $p>1$

Claim: $\frac{1}{n^p \ln(n)} \overset{?}{<} \frac{1}{n^p}$

$n^p \overset{?}{<} n^p \ln(n)$

$1 < \ln(n) \Rightarrow$ true for any $n \geq 3$

$\Rightarrow$ Since $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p>1$ and we showed that $\frac{1}{n^p \ln n} < \frac{1}{n^p}$, we conclude that $\sum_{n=2}^{\infty} \frac{1}{(\ln n) n^p}$ converges for $p>1$.

Note: We showed that $\frac{1}{(\ln n) n^p} < \frac{1}{n^p}$ only for $n \geq 3$ and $\sum^{\infty} a_n$ starts at $n=2$, but we know that adding finite terms to a converging series does not affect convergence. Thus $\sum_{n=2}^{\infty} a_n$ converges for $p>1$ as well.

Any other p's? NO.

$p=1$: $\sum_{n=2}^{\infty} \frac{1}{n \cdot \ln n}$: use integral test to show divergence

$$\Rightarrow \int_2^{\infty} \frac{1}{x \ln x} = \lim_{t\to\infty} \int_2^t \frac{1}{x \ln x}\, dx \quad : \text{let } u = \ln \cancel{x},\ du \cancel{x} = dx$$

$$\Rightarrow \lim_{t\to\infty} \int_{x=2}^{x=t} \frac{du}{u} = \lim_{t\to\infty} \int_{x=2}^{x=t} \frac{1}{u}\, du = \lim_{t\to\infty} \left(\ln(\ln(x)) \Big|_2^t \right) = \lim_{t\to\infty} \left(\ln(\ln(t)) - \ln(\ln(2)) \right) = \infty$$

: conclusion: $p=1$, $\sum_{n=2}^{\infty} \frac{1}{(\ln n) n^p}$ diverges

$p<1$: $\sum_{n=1}^{\infty} \frac{1}{n^p (\ln n)}$: compare with $\sum_{n=1}^{\infty} \frac{1}{n (\ln n)}$ to show divergence.

$\frac{1}{n^p \ln(n)} > \frac{1}{n \ln(n)}$; for $p<1$

$\Rightarrow$ by comparing the two, we conclude divergence