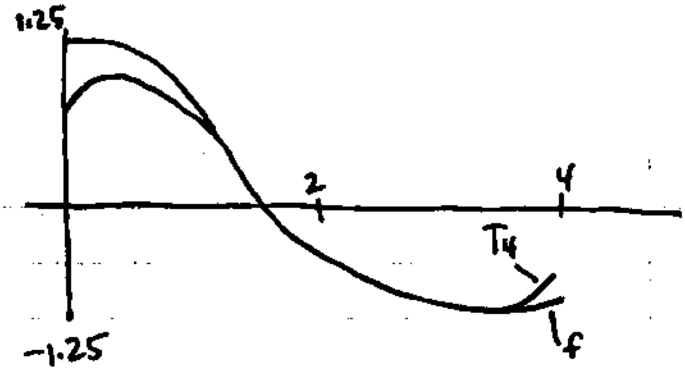
§11.12

(4), 6, 14, 16, 26, 28

6.// $f(x) = \cos x \quad a = 2\pi/3 \quad n = 4$

n	$f^{(n)}(x)$	$f^{(n)}(2\pi/3)$
0	$\cos x$	$-1/2$
1	$-\sin x$	$-\sqrt{3}/2$
2	$-\cos x$	$1/2$
3	$\sin x$	$\sqrt{3}/2$
4	$\cos x$	$-1/2$



$$T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}(2\pi/3)}{n!}\left(x - \frac{2\pi}{3}\right)^n = \boxed{-\frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{2\pi}{3}\right) + \frac{1}{4}\left(x - \frac{2\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{2\pi}{3}\right)^3 - \frac{1}{48}\left(x - \frac{2\pi}{3}\right)^4}$$

14.// $f(x) = x^{-2} \quad a = 1 \quad n = 2 \quad 0.9 \leq x \leq 1.1$

a.//

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6

$$T_2(x) = 1 - 2(x-1) + \frac{6}{2!}(x-1)^2$$

$$\boxed{T_2(x) = 1 - 2(x-1) + 3(x-1)^2}$$

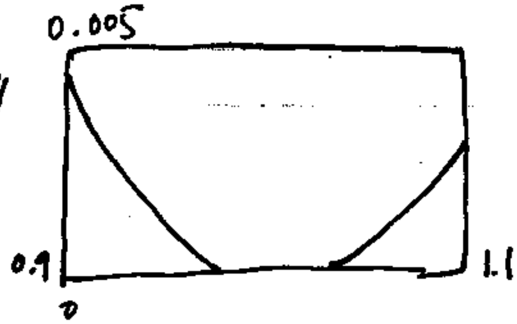
b.// $|R_2(x)| \leq \frac{M}{3!}|x-1|^3$ where $|f'''(x)| \leq M$

Now $0.9 \leq x \leq 1.1 \Rightarrow |x-1| \leq 0.1 \Rightarrow |x-1|^3 \leq 0.001$
Since $f'''(x)$ is decreasing on $[0.9, 1.1]$ we can take
$M = |f'''(0.9)| = 24/(0.9)^5$

so $|R_2(x)| \leq \frac{24/(0.9)^5}{6}(0.001) \approx \boxed{0.00677404}$

$\longrightarrow$

c.//



So on $[0.9, 1.1]$, $|R_2(x)|$ seems less than 0.005 (close approximation in part (b))

16.// a.// $f(x) = \cos x \quad a = \pi/3 \quad n = 4 \quad 0 \le x \le 2\pi/3$

n	$f^{(n)}(x)$	$f^{(n)}(\pi/3)$
0	$\cos x$	$1/2$
1	$-\sin x$	$-\sqrt{3}/2$
2	$-\cos x$	$-1/2$
3	$\sin x$	$\sqrt{3}/2$
4	$\cos x$	$1/2$

$$\boxed{T_4(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{48}\left(x - \frac{\pi}{3}\right)^4}$$

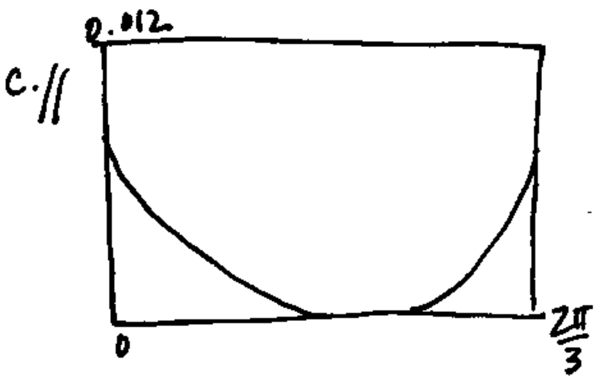
b.// $|R_4(x)| \le \frac{M}{5!}\left|x - \frac{\pi}{3}\right|^5$, where $|f^{(5)}(x)| \le M$

Now $0 \le x \le \frac{2\pi}{3} \Rightarrow (x - \frac{\pi}{3})^5 \le (\frac{\pi}{3})^5$

$|\cos^{(n)}(x)| \le 1$ so let $M = 1$

So $|R_4(x)| \le \frac{1}{5!}\left(\frac{\pi}{3}\right)^5 \approx \boxed{0.0105}$

c.//



So on $[0, 2\pi/3]$

$|R_4(x)|$ is less than 0.01 (close approximation in part (b))

26.// We know the MacLaurin series for $\ln(1+x)$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

This is an alternating series! So we can use the Alternating Series Estimation theorem which says the error is <u>less</u> <u>than</u> <u>the</u> <u>first</u> <u>neglected</u> <u>term</u>

$$|a_6| = \frac{(0.4)^6}{6} \approx 0.0007 < 0.001$$

So we need the $\boxed{\text{first five non-zero terms.}}$

28.// We know $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

By the Alternating Series Estimation Theorem, the error is less than $\left|-\frac{1}{6!}x^6\right|$

To ensure the desired accuracy, we set this < 0.005

$$\left|-\frac{1}{6!}x^6\right| < 0.005 \rightarrow x^6 < 3.6 \rightarrow \boxed{|x| \leq 1.238}$$

Check graphically:

→ $x = 1.244$ is the intersection so when $x < 1.244$ our approximation falls within the desired range. Since the cosine and our approximation are even functions we know the same is true for $x > -1.244$.

Optional:

(4.) $f(x) = e^x \quad a=2 \quad n=3$

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	e^x	e^2
1	e^x	e^2
2	e^x	e^2
3	e^x	e^2

$$\boxed{T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(2)}{n!}(x-2)^n = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3}$$