# § 11.11

(2), 6, 8, 12, 18

6.// $\frac{1}{\sqrt[5]{32-x}}$

$$\frac{1}{\sqrt[5]{32-x}} = \frac{1}{\sqrt[5]{32(1-\frac{x}{32})}} = \frac{1}{2\sqrt[5]{1-\frac{x}{32}}} = \frac{1}{2}\left(1-\frac{x}{32}\right)^{-1/5}$$

by binomial series expansion $= \frac{1}{2}\sum_{n=0}^{\infty}\binom{-1/5}{n}\left(-\frac{x}{32}\right)^n$

$$= \frac{1}{2}\left[1+\left(-\tfrac{1}{5}\right)\left(-\frac{x}{2^5}\right)+\frac{\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)}{2!}\frac{x^2}{2^{10}}+\frac{\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)\left(-\frac{11}{5}\right)}{3!}\left(-\frac{x^3}{2^{15}}\right)+\cdots\right]$$

$$\boxed{= \frac{1}{2}+\frac{1}{5\cdot 2^6}x+\frac{1\cdot 6}{5^2\cdot 2!\cdot 2^{11}}x^2+\frac{1\cdot 6\cdot 11}{5^3\cdot 3!\cdot 2^{16}}x^3+\cdots}$$

Converges for $\left|-\frac{x}{32}\right|<1 \rightarrow |x|<32$ so $\boxed{\text{radius of conv.} = 32}$

8.// $\frac{x^2}{\sqrt{2+x}}$

$$\frac{x^2}{\sqrt{2+x}} = \frac{x^2}{\sqrt{2(1+\frac{x}{2})}} = \frac{x^2}{\sqrt{2}\sqrt{1+\frac{x}{2}}} = \frac{x^2}{\sqrt{2}}\left(1+\frac{x}{2}\right)^{-1/2}$$

by binomial series expansion $= \frac{x^2}{\sqrt{2}}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\frac{x}{2}\right)^n$

$$= \frac{x^2}{\sqrt{2}}\left[1+\left(-\tfrac{1}{2}\right)\left(\frac{x}{2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{2}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x}{2}\right)^3+\cdots\right]$$

$$\boxed{= \frac{x^2}{\sqrt{2}}+\frac{x^2}{\sqrt{2}}\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1)}{n!\,2^{2n}}x^n}$$

Converges for $\left|\frac{x}{2}\right|<1 \rightarrow |x|<2$ so $\boxed{\text{Radius of conv.} = 2}$

12.// a.// $\frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^{2n}$

$$\boxed{= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)\, x^{2n}}{2^n \cdot n!}}$$

b.// We know $\sinh^{-1} x = \int \frac{1}{\sqrt{1+x^2}}\,dx$

So we integrate the answer to part(a):

$$= C + x + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)\, x^{2n+1}}{2^n \cdot n! \cdot (2n+1)}$$

We know $\sinh^{-1}(0) = 0$ so $C$ must $= 0$

$$\boxed{\therefore \sinh^{-1} x = x + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)\, x^{2n+1}}{2^n \cdot n! \cdot (2n+1)}}$$

18.// a.// $f(x) = \frac{1}{\sqrt{1+x^3}}$

$$= (1+x^3)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (x^3)^n$$

$$= 1 + (-\tfrac{1}{2})(x^3) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(x^3)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(x^3)^3 + \cdots$$

$$\boxed{= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)\, x^{3n}}{2^n \cdot n!}}$$

b.// $f^{(9)}(0) = ?$

In our typical formula for a Maclaurin series, the coefficient of the $x^9$ term is $\frac{f^{(9)}(0)}{9!}$. We therefore compare this with the coefficient of the $x^9$ term in our series found in part (a): $\frac{(-1)^3 \cdot 1 \cdot 3 \cdot 5}{2^3 \cdot 3!}$

$$\frac{(-1)^3 \cdot 1 \cdot 3 \cdot 5}{2^3 \cdot 3!} = -\frac{5}{16} = \frac{f^{(9)}(0)}{9!} \quad \therefore \quad f^{(9)}(0) = -\frac{5}{16}(9!) = \boxed{-113{,}400}$$

Optional:

(2.) $$\frac{1}{(1+x)^4} = (1+x)^{-4} = \sum_{n=0}^{\infty} \binom{-4}{n} x^n$$

$$\binom{-4}{n} = \frac{(-4)(-5)(-6)\cdots(-4-n+1)}{n!} = \frac{(-1)^n \cdot 2\cdot 3\cdot 4\cdot 5 \cdots (n+1)(n+2)(n+3)}{2\cdot 3\cdot n!}$$

$$= \frac{(-1)^n (n+1)(n+2)(n+3)}{6}$$

$$\boxed{\text{So } \frac{1}{(1+x)^4} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)(n+3)}{6} x^n}$$

$|x| < 1$ so $\boxed{\text{Radius of convergence is } 1}$