A SUMMARY OF CONVERGENCE TESTS

All the tests in this handout are stated for $\sum_{n=0}^{\infty} a_n$, with n starting at 0. Remember: the first few terms of the series don't matter in deciding whether it converges, so the tests will apply just as well to $\sum_{n=29}^{\infty} a_n$, for example.

1. Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.
Try this one first to weed out the stupid examples!

2. Standard Test Series

1. **Geometric Series**:
$$\sum_{n=0}^{\infty} q^n = \begin{cases} \frac{1}{1-q}, & \text{if } |q| < 1 \\ \text{diverges}, & \text{if } |q| \geq 1. \end{cases}$$
2. p-**Series**: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$; diverges for $p \leq 1$

3. Plain Comparison Tests

Let $\sum a_n, \sum b_n$ be two series with *positive terms*, that is, $a_n \geq 0, b_n \geq 0$. Then

1. If $a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ also converges.
2. If $a_n \geq b_n$ and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

4. Limit Comparison Test

Let $\sum a_n, \sum b_n$ be two series with *positive terms*, that is, $a_n \geq 0, b_n \geq 0$. Form the limit
$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$
If L is *finite and non-zero* (i.e. $0 < L < \infty$), then $\sum a_n$ converges if $\sum b_n$ converges, and $\sum a_n$ diverges if $\sum b_n$ diverges.
So, you find a simpler series b_n to compare a_n to. Rule of thumb: If you're given a series $\sum a_n$ with a_n a rational function in n, for example
$$a_n = \frac{n^5 - 36n^3 + 1}{n^7 - n^6 + n^5 - n^4 + n^3 - n^2 + n - 1},$$
a good series to compare it to is the series given by the quotient of the highest powers of n in the numerator and the denominator, in our example
$$\frac{n^5}{n^7} = \frac{1}{n^2}$$

The limit comparison test is more powerful than the plain one – use it if you can't find an obvious comparison series which is always bigger or smaller than the one you're testing.

5. Integral Test

Suppose that $a_n = f(n)$, where f is a function which is *continuous, positive and decreasing* on the interval $[1, \infty)$. Then $\sum a_n$ converges or diverges depending on whether the improper integral

$$\int_1^\infty f(x)dx$$

converges or diverges.

DO NOT try this test if there is $n!$ in a_n – we don't have a nice function whose values at the integers are the factorials.

6. Alternating Series Test

A series of the form $\sum_{n=0}^{\infty}(-1)^n a_n = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \dots$ with all $a_n \geq 0$ is called an *alternating* series (since the signs in it alternate).

The Test: Let $\sum(-1)^n a_n$, all $a_n \geq 0$, be an alternating series. If the following two conditions hold:

1. **$\{a_n\}$ is decreasing, i.e. $a_1 \geq a_2 \geq a_3 \geq \dots$,**
2. $\lim_{n\to\infty} a_n = 0$,

then the series $\sum(-1)^n a_n$ converges.

7. Absolute and Conditional Convergence

Let $\sum_{n=0}^{\infty} a_n$ be a series whose terms may be positive or negative. $\sum_{n=0}^{\infty} a_n$ is said to be *absolutely convergent* if the series of absolute values $\sum_{n=0}^{\infty} |a_n|$ converges.

Theorem: If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then it's convergent.

A series $\sum_{n=0}^{\infty} a_n$ is *conditionally convergent* if it's convergent but not absolutely convergent.

Example: $\sum \frac{(-1)^n}{n}$ is convergent by the Alternating Series Test. Its series of absulute values, $\sum \frac{1}{n}$, diverges by the p-series test. Therefore, $\sum \frac{(-1)^n}{n}$ is convergent, but not absolutely convergent: we say it's *conditionally convergent.*

8. Ratio Test for Absolute Convergence

Let $\sum a_n$ be a series. Form the limit

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}.$$

1. If $L < 1$, $\sum a_n$ conveges absolutely;
2. If $L > 1$, $\sum a_n$ diverges;
3. If $L = 1$, the test gives no information.

9. Root Test for Absolute Convergence

Let $\sum a_n$ be a series. Form the limit

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

1. If $L < 1$, $\sum a_n$ conveges absolutely;
2. If $L > 1$, $\sum a_n$ diverges;
3. If $L = 1$, the test gives no information.

Note: if $\sum a_n$ is a series with positive terms, then it's equal to its series of absolute values, so the Root and Ratio Tests give us another tool for testing convergence of positive series.

10. Maclaurin Series of Familiar Functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ for } -\infty < x < \infty$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \text{ for } -1 \le x < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \text{ for } -1 < x \le 1$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \text{ for } -\infty < x < \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \text{ for } -\infty < x < \infty$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \text{ for } -1 \le x \le 1$$

$$(1+x)^t = \sum_{n=0}^{\infty} \binom{t}{n} x^n, \text{ for } -1 < x < 1, \text{ where } \binom{t}{n} = \frac{t(t-1)\dots(t-n+1)}{n!}$$

Pay attention to the interval of convergence!