Name: ______________________ ID#: ______________

Solutions to Midterm I

Math S-1ab
Calculus I and II
Summer 2004

July 12, 2004

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

> *A student suspected of academic dishonesty in any form is subject to review and disciplinary action by the Summer School Administrative Board. Disciplinary action may include, but is not limited to, required withdrawal from the course and/or required withdrawal from the Summer School.* —*Handbook for Students*

Problem Number	Possible Points	Points Earned
1	20	
2	45	
3	10	
4	10	
5	15	
Total	100	

1. (20 Points) *Find the limits of the following functions, if they exist. If the limit does not exist, say so.*

(i) $\lim_{x\to 4} \frac{\sqrt{x}-4}{x-4}$

Solution. The numerator approaches -2, and the denominator approaches 0. We have

$$\lim_{x\to 4^-} \frac{\sqrt{x}-4}{x-4} = -\infty; \qquad \lim_{x\to 4^+} \frac{\sqrt{x}-4}{x-4} = +\infty.$$

The limit does not exist. □

(ii) $\lim_{x\to 0} \frac{2-\cos x}{x^2+1}$

Solution. The numerator approaches 1 and the denominator approaches 2. The limit is therefore $\frac{1}{2}$. □

(iii) $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$

Solution. As discussed in class and in the book, this limit does not exist. For there are points x arbitrarily close to 0 such that $\sin\left(\frac{1}{x}\right) = 1$, and points x arbitrarily close to 0 such that $\sin\left(\frac{1}{x}\right) = -1$. □

(iv) $\lim_{x\to\infty} \frac{1}{x}\cos x$

Solution. We have

$$-\frac{1}{x} \leq \frac{1}{x}\cos x \leq \frac{1}{x}.$$

The left- and right-hand terms in this threefold inequality both go to 0 as $x \to \infty$. Hence by the Squeeze Theorem the limit is ∞. □

(v) $\lim_{x\to\infty} \frac{\sqrt{x^2+x}}{\sqrt{3x^2-x}}$.

Solution. We may factor out the largest power of x:

$$\begin{aligned} \frac{\sqrt{x^2+x}}{\sqrt{3x^2-x}} &= \frac{\sqrt{x^2(1+1/x)}}{\sqrt{x^2(3-1/x}} \\ &= \frac{\sqrt{1+1/x}}{\sqrt{3-1/x}}. \end{aligned}$$

As $x \to \infty$, the numerator approaches $\sqrt{1}$ and the denominator approaches $\sqrt{3}$. Hence the limit is $\dfrac{1}{\sqrt{3}}$. $\square$

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2. (45 Points) *Find the following derivatives:*

(i) $\frac{d}{dx}\left(x^4 + 3x^3 + x + 2\right)$

Solution. By the power rule,

$$\frac{dy}{dx} = 4x^3 + 9x^2 + 1.$$

□

(ii) $\frac{d}{d\theta}\frac{3}{1+\cos\theta}$

Solution.

$$\frac{d}{d\theta}3(1+\cos\theta)^{-1} = 3(1+\cos\theta)^{-2}(-\sin\theta) = -\frac{3\sin\theta}{(1+\cos\theta)^2}$$

□

(iii) $\frac{d}{dx}\sin\left(\frac{\pi}{5}\right)$

Solution. $\sin\left(\frac{\pi}{5}\right)$ is a *constant* with respect to x, and hence the derivative is zero. □

(iv) f', *where* $f(t) = e^{2t}\sqrt{1+t^2}$

Solution. Using the product rule, we have

$$\begin{aligned} f'(t) &= e^{2t}(2)\sqrt{1+t^2} + e^{2t}(\tfrac{1}{2})(1+t^2)^{-1/2}(2t) \\ &= e^{2t}\frac{2t^2+t+2}{\sqrt{1+t^2}} \end{aligned}$$

□

(v) g', *where* $g(x) = \frac{\sin x}{x}$

Solution. Using the quotient rule, we have

$$g'(x) = \frac{x\cos x - \sin x}{x^2}.$$

□

(vi) h', *where* $h(s) = \ln(\ln s)$

Solution.

$$h'(s) = \frac{1}{\ln s}\frac{1}{s} = \frac{1}{s\ln s}.$$

□

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3. (10 Points) *Find $\sqrt[3]{65}$ using a linear approximation.*

Solution. Let $f(x) = \sqrt[3]{x}$. We know $f(64) = 4$ and we'd like to know $f(65)$. By a linear approximation,

$$f(65) \approx f(64) + f'(64) = 4 + \frac{1}{3 \cdot 4^2} = 4 + \frac{1}{48} = \frac{193}{48}.$$

$\square$

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4. (10 Points) *Prove that right now there are two points on the equator on opposite sides of the world that have the exact temperature.*

Solution. Let t be the temperature function at the equator. We have to choose some coordinates, so let $t(x)$ be the temperature which is at x degrees longitude and 0 degrees latitude (the equator). West longitudes are considered negative and East ones positive. Hence the domain of t is $[-180, 180]$ and

$$(-180) = t(180)$$

since these are both the same point (at the intersection of the international date line and equator).

Let f be the function whose domain is $[0, 180]$ and take x to $t(x)-t(x-180)$. This function is the difference between any point in the eastern hemisphere along the equator and its antipode on the other side of the world. Then

$$f(0) = t(0) - t(-180)$$

and

$$f(180) = t(180) - t(0) = -f(0).$$

If $f(0) > 0$, $f(180) < 0$, and there is a point x in between 0 and 180 such that $f(x) = 0$, or $t(x) = t(x-180)$. On the other hand, if $f(0) < 0$, then $f(180) > 0$, and the same thing works. On the third hand, if $f(0) = 0$, we are done because then $t(0) = t(180)$.

The only thing we're really assuming is that temperature is continuously distributed over the Earth. Since the equator is a closed path, it follows that the temperature function going around the equator is periodic. $\square$

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5. (15 Points) *Define two new functions*

$$\cosh x = \frac{e^x + e^{-x}}{2}; \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

(a) Prove that

$$\frac{d}{dx}\cosh x = \sinh x; \qquad \frac{d}{dx}\sinh x = \cosh x.$$

Solution. This follows from the chain rule. □

(b) Let $\sinh^{-1} x$ *be the inverse function to* sinh. *Prove*

$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}.$$

Solution. Notice that

$$\cosh^2 y - \sinh^2 y = 1,$$

for all y, as can be checked. If $y = \sinh^{-1}(x)$, then $\sinh y = x$, and we have by implicit differentiation

$$\cosh y \frac{dy}{dx} = 1; \frac{dy}{dx} \qquad = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}.$$

□

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