

Name: _____ ID#: _____

Solutions to Midterm II

Math S-1ab
Calculus I and II
Summer 2004

July 23, 2004

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

A student suspected of academic dishonesty in any form is subject to review and disciplinary action by the Summer School Administrative Board. Disciplinary action may include, but is not limited to, required withdrawal from the course and/or required withdrawal from the Summer School.

—Handbook for Students

Problem Number	Possible Points	Points Earned
1	10	
2	25	
3	15	
4	10	
5	15	
6	15	
7	10	
Total	100	

1. (10 Points) Gravel is being dumped from a conveyor belt at a rate of 30 cubic feet per minute and its coarseness is such that it forms a pile in the shape of a cone whose base is always twice its height. How fast is the height of the pile increasing when the pile is 10 feet high?

Solution We know that the volume of the cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius, which in this case is h (the base is twice the radius, and in this cone it's also twice the height). So

$$V = \frac{1}{3}\pi h^3;$$
$$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt}.$$

When $h = 10$, we have

$$30 = \pi(10)^2 \frac{dh}{dt};$$
$$\frac{dh}{dt} = \frac{3}{10\pi} \frac{\text{ft}^3}{\text{min}}.$$

□

The grading schema for this problem was:

- 4 points for computing V in terms of h ;
- 4 points for computing $\frac{dh}{dt}$ in terms of h ;
- 2 points for evaluating $\frac{dh}{dt}$ when $h = 10$ and getting a number.

The average score on this problem was 8.82 / 10.

2. (25 Points) Consider the function

$$f(x) = x^2 - 2x + 6 \ln|x + 2|.$$

(i) What is the domain of f ?

Solution Thanks to the absolute value, the log term has a value for all $x \neq -2$. Hence the domain of f is $\mathbb{R} \setminus \{-2\}$. This part was worth 2 points.

(ii) What asymptotes does the graph have, if any?

Solution The only point at which there could be an asymptote is $x = -2$, and since

$$\lim_{x \rightarrow -2} \ln|x + 2| = -\infty,$$

there is a vertical asymptote. Since $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow -\infty} f(x) = \infty$, there are no horizontal asymptotes. This part was worth 3 points, and everybody missed 2 for not checking for horizontal asymptotes.

(iii) On what intervals is the function increasing? decreasing?

Solution Oh, boy. I tried to work the numbers out just right and made a mistake (twice). We have

$$\begin{aligned} f'(x) &= 2x - 2 + \frac{6}{x + 2} \\ &= \frac{2(x - 1)(x + 2) + 6}{x + 2} \\ &= \frac{2(x^2 + x + 1)}{x + 2}. \end{aligned}$$

The discriminant of the numerator is $1^2 - 4(1)(1) = -3 < 0$, so this quadratic has no real roots. Hence the derivative is *never* zero. So the function is increasing over its entire domain: $(-\infty, -2) \cup (-2, \infty)$.

This part was worth 5 points, of which 3 was for computing the derivative correctly.

(iv) What is/are the critical point(s)? Give their x - and y -values. Label them as local or global maxima or minima.

Solution No critical points, so no maxima or minima. It is true that f' is undefined at -2 , but this does not make -2 a critical point because -2 is *not in the domain of f* .

This part was worth 2 points.

(v) On what intervals is the function concave up? concave down?

Solution Luckily, the second derivative works out (somewhat) better:

$$\begin{aligned} f''(x) &= 2 - \frac{6}{(x+2)^2} \\ &= \frac{2(x+2)^2 - 6}{(x+2)^2} \\ &= \frac{2(x^2 + 4x + 1)}{(x+2)^2}. \end{aligned}$$

This time the numerator has roots $-2 \pm \sqrt{3}$. The denominator is always positive, except at -2 , where it is zero. Thus f'' is positive (and f is concave up) on $(-\infty, -2 - \sqrt{3}) \cup (-2 + \sqrt{3}, \infty)$; and f'' is negative (and f is concave down) on $(-2 - \sqrt{3}, -2 + \sqrt{3})$.

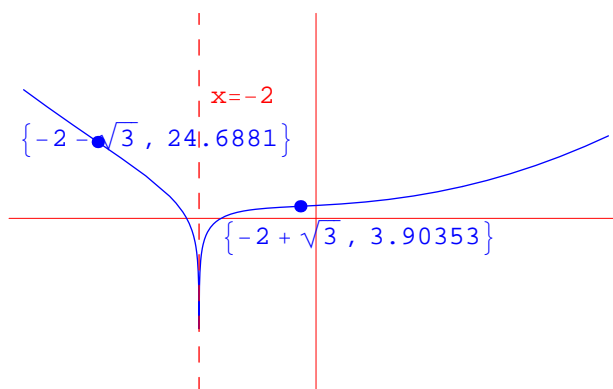
This part was worth 5 points, of which 3 were for computing the second derivative correctly.

- (vi) Where are the inflection points, if any? Only give x -values since the y -values are quite complicated.

Solution (2 points) The inflection points are at $-2 \pm \sqrt{3}$.

- (vii) Sketch the graph of f , labeling the important points.

Solution (Up to 6 points) Here is a Mathematica graph:



□

My errors in algebra caused this problem to be much harder than I intended. However, they did not make the problem undoable. I tried to grade the problem as fairly as possible, keeping in mind that students couldn't attempt a problem that relied on intermediate information if they didn't get the intermediate information. This meant that the problem was actually worth a varying amount of points, then graded on that scale, then scaled to something over 25 points.

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For instance, part (vii) (the graph) was worth 2 points if part (ii) was answered, plus 2 points if part (iii) was answered, plus 2 points if part (v) was answered.

There is less “intermediate” information than you might think. You can get most of (iii)–(vi) just by computing derivatives, which most of you did correctly. Interpreting them was slightly harder.

In summary, I admit this problem was harder than intended, but maintain that it was fairly posed and fairly graded. I won’t disregard the scores, but I will promise that there will be another (easier) graphing question on the final.

The average score on this problem was 15.13 / 25.

3. (15 Points) Find the following limits.

$$(i) \lim_{x \rightarrow 0^+} \frac{\tan x}{1 - \cos x}$$

Solution Since this limit is of the form $\frac{0}{0}$, we can use L'Hôpital's Rule.

$$\lim_{x \rightarrow 0^+} \frac{\tan x}{1 - \cos x} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{\sec^2 x}{\sin x}.$$

Now the denominator is positive and approaches zero, and the numerator approaches 1. Hence the quotient approaches $+\infty$.

$$(ii) \lim_{x \rightarrow 0} \frac{x}{\cos x}$$

Solution The numerator approaches zero while the denominator approaches 1. Hence the limit is 1. This was a trap to make sure you remember to check the hypotheses of L'Hôpital's Rule before you apply it!

$$(iii) \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \frac{1}{2}x - 1}{x^2}$$

Solution We have to apply L'Hôpital's Rule twice.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \frac{1}{2}x - 1}{x^2} &\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\frac{1}{2}(1+x)^{-1/2} - \frac{1}{2}}{2x} \\ &\stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-\frac{1}{4}(1+x)^{-3/2}}{2} = -\frac{1}{8}. \end{aligned}$$

□

Each of these problems was worth 5 points. In the future, please do not abuse the equals sign. For instance, on part (iii), many of you wrote something like:

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \frac{1}{2}x - 1}{x^2} = \frac{\frac{1}{2}(1+x)^{-1/2} - \frac{1}{2}}{2x} = \frac{-\frac{1}{4}(1+x)^{-3/2}}{2} = -\frac{1}{8}.$$

All of these equals signs are wrong because of the missing limit operator. I understand that on an exam, speed is of the essence, but we math teachers get our hackles raised when we see errors in grammar like this.

The average score on this problem was 12.45 / 15.

4. (10 Points) Find the slope of the line tangent to the curve

$$y^2 = x^3 + x$$

at the point $(1, \sqrt{2})$.

Solution We differentiate the relation implicitly, regarding y as a function of x .

$$2y \frac{dy}{dx} = 3x^2 + 1.$$

At our particular point we have

$$\frac{dy}{dx} = \frac{3x^2 + 1}{2y} = \frac{4}{2\sqrt{2}} = \sqrt{2}.$$

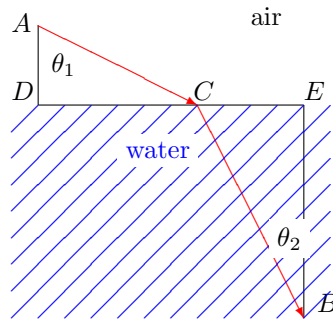
For those that differentiated the relation implicitly, 6 points were for doing that correctly and 2 for solving for $\frac{dy}{dx}$ in terms of x and y . For those that solved for y first and then differentiated, 2 points were for doing that correctly and 6 were for computing $\frac{dy}{dx}$. The remaining 2 points were for evaluating the derivative at the right point.

The average score on this problem was 8.91 / 10.

5. (15 Points) Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to Fermat's principle, a ray of light will travel from a point A to a point B in the water by a path ACB that minimizes the time taken. Show that

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2},$$

where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are as shown. This equation is known as *Snell's Law*.



Hint: Let x be the distance DC , and w the distance DE . What x minimizes the time taken?

Solution Following the hint, we will compute the time taken for the light to travel from A to B in terms of x .

Let $a = AD$ and $b = BE$. Then the distance traveled is $\sqrt{a^2 + x^2} + \sqrt{a^2 + (w - x)^2}$ (the two hypotenuses), and the time it takes to travel this distance is

$$t(x) = \frac{1}{v_1} \sqrt{a^2 + x^2} + \frac{1}{v_2} \sqrt{a^2 + (w - x)^2}.$$

Therefore

$$\begin{aligned} \frac{dt}{dx} &= \frac{x}{v_1 \sqrt{a^2 + x^2}} - \frac{w - x}{v_2 \sqrt{a^2 + (w - x)^2}} \\ &= \frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2}. \end{aligned}$$

As usual, the time is minimized when $\frac{dt}{dx} = 0$, which means

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}.$$

□

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I tried to give 5 points for computing the time relation correctly, 5 points for differentiating it, and the remaining 5 for rearranging to get the desired answer.

The average score on this problem was 3.45 / 15. I was somewhat disappointed. I understand that Problem 2 may have sucked time away from problems like this one, however. I will also make sure an optimization problem appears on the final.

6. (15 Points) Find the following indefinite integrals:

(i) $\int (x^2 + 3x + 5) dx$

Solution The power rule in reverse:

$$\int (x^2 + 3x + 5) dx = \frac{x^3}{3} + \frac{3x^2}{2} + 5x + C.$$

(ii) $\int \frac{3}{\sqrt{1-x^2}} dx$

Solution

$$\int \frac{3}{\sqrt{1-x^2}} dx = 3 \arcsin(x) + C.$$

(iii) $\int \frac{17}{x} dx$

Solution

$$\int \frac{17}{x} dx = 17 \ln |x| + C.$$

Each part was worth 5 points, and most did quite well on this problem. I did not take points off for missing “+C”s. I did take off points for forgetting the absolute value bars in (iii). Without it, the indefinite integral has a much smaller domain.

The average score on this problem was 12.36 / 15.

7. (10 Points) Find $\int_0^1 4x(x^2 + 1)^9 dx$.

Solution We let $u = x^2$, so $du = 2x dx$. Then

$$\begin{aligned}\int_0^1 4x(x^2 + 1)^9 dx &= \int_1^2 2u^9 du \\ &= \frac{u^{10}}{5} \Big|_1^2 \\ &= \frac{1024 - 1}{5} = \frac{1023}{5}.\end{aligned}$$

4 points were for guessing the right substitution, 3 were for correctly transforming the integral (this includes the limits), and 3 were for evaluating and simplifying correctly.

The average score on this problem was 8.45 / 10.

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