

Math 1a Homework Solutions

Section 4.5

Note: The use of l'Hospital's Rule is indicated by an H above the equal sign: $\stackrel{H}{=}$

1. (a) $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$.

(b) $\lim_{x \rightarrow a} \frac{f(x)}{p(x)} = 0$ because the numerator approaches 0 while the denominator becomes large.

(c) $\lim_{x \rightarrow a} \frac{h(x)}{p(x)} = 0$ because the numerator approaches a finite number while the denominator becomes large.

(d) If $\lim_{x \rightarrow a} p(x) = \infty$ and $f(x) \rightarrow 0$ through positive values, then $\lim_{x \rightarrow a} \frac{p(x)}{f(x)} = \infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x^2$.] If $f(x) \rightarrow 0$ through negative values, then $\lim_{x \rightarrow a} \frac{p(x)}{f(x)} = -\infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = -x^2$.] If $f(x) \rightarrow 0$ through both positive and negative values, then the limit might not exist. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x$.] It is not possible to evaluate this limit.

(e) $\lim_{x \rightarrow a} \frac{p(x)}{q(x)}$ is an indeterminate form of type $\frac{\infty}{\infty}$.

2. (a) $\lim_{x \rightarrow a} [f(x)p(x)]$ is an indeterminate form of type $0 \cdot \infty$.

(b) When x is near a , $p(x)$ is large and $h(x)$ is near 1, so $h(x)p(x)$ is large. Thus, $\lim_{x \rightarrow a} [h(x)p(x)] = \infty$.

(c) When x is near a , $p(x)$ and $q(x)$ are both large, so $p(x)q(x)$ is large. Thus, $\lim_{x \rightarrow a} [p(x)q(x)] = \infty$.

4. (a) $\lim_{x \rightarrow a} [f(x)]^{g(x)}$ is an indeterminate form of type 0^0 .

(b) If $y = [f(x)]^{p(x)}$, then $\ln y = p(x) \ln f(x)$. When x is near a , $p(x) \rightarrow \infty$ and $\ln f(x) \rightarrow -\infty$, so $\ln y \rightarrow -\infty$. Therefore, $\lim_{x \rightarrow a} [f(x)]^{p(x)} = \lim_{x \rightarrow a} y = \lim_{x \rightarrow a} e^{\ln y} = 0$, provided f^p is defined.

(c) $\lim_{x \rightarrow a} [h(x)]^{p(x)}$ is an indeterminate form of type 1^∞ .

(d) $\lim_{x \rightarrow a} [p(x)]^{f(x)}$ is an indeterminate form of type ∞^0 .

(e) If $y = [p(x)]^{q(x)}$, then $\ln y = q(x) \ln p(x)$. When x is near a , $q(x) \rightarrow \infty$ and $\ln p(x) \rightarrow \infty$, so $\ln y \rightarrow \infty$. Therefore, $\lim_{x \rightarrow a} [p(x)]^{q(x)} = \lim_{x \rightarrow a} y = \lim_{x \rightarrow a} e^{\ln y} = \infty$.

(f) $\lim_{x \rightarrow a} \sqrt[q(x)]{p(x)} = \lim_{x \rightarrow a} [p(x)]^{1/q(x)}$ is an indeterminate form of type ∞^0 .

14. $\lim_{x \rightarrow \infty} \frac{e^x}{x^3} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{3x^2} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{6x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{6} = \infty$

30. $\lim_{x \rightarrow 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) = \lim_{x \rightarrow 1} \frac{x-1 - \ln x}{(x-1) \ln x} \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{1 - 1/x}{(x-1)(1/x) + \ln x} \cdot \frac{x}{x}$
 $= \lim_{x \rightarrow 1} \frac{x-1}{x-1 + x \ln x} \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{1}{1 + 1 + \ln x} = \frac{1}{2+0} = \frac{1}{2}$

34. $y = \left(1 + \frac{a}{x}\right)^{bx} \Rightarrow \ln y = bx \ln \left(1 + \frac{a}{x}\right)$, so

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{b \ln(1 + a/x)}{1/x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{b \left(\frac{1}{1 + a/x} \right) \left(-\frac{a}{x^2} \right)}{-1/x^2} = \lim_{x \rightarrow \infty} \frac{ab}{1 + a/x} = ab \Rightarrow$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^{bx} = \lim_{x \rightarrow \infty} e^{\ln y} = e^{ab}.$$

$$51. \lim_{x \rightarrow \infty} \frac{e^x}{x^n} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{nx^{n-1}} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{n(n-1)x^{n-2}} \stackrel{H}{=} \dots \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{e^x}{n!} = \infty$$

$$52. \lim_{x \rightarrow \infty} \frac{\ln x}{x^p} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/x}{px^{p-1}} = \lim_{x \rightarrow \infty} \frac{1}{px^p} = 0 \text{ since } p > 0.$$

$$54. \text{ (a) } \lim_{t \rightarrow \infty} v = \lim_{t \rightarrow \infty} \frac{mg}{c} (1 - e^{-ct/m}) = \frac{mg}{c} \lim_{t \rightarrow \infty} (1 - e^{-ct/m}) \\ = \frac{mg}{c} (1 - 0) \quad [\text{because } -ct/m \rightarrow -\infty \text{ as } t \rightarrow \infty] = \frac{mg}{c},$$

which is the speed the object approaches as time goes on, the so-called limiting velocity.

$$\text{(b) } \lim_{m \rightarrow \infty} v = \lim_{m \rightarrow \infty} \frac{mg}{c} (1 - e^{-ct/m}) = \frac{g}{c} \lim_{m \rightarrow \infty} \frac{1 - e^{-ct/m}}{1/m} \stackrel{H}{=} \frac{g}{c} \lim_{m \rightarrow \infty} \frac{-e^{-ct/m} (ct/m^2)}{-1/m^2} \\ = \frac{g}{c} (ct) \lim_{m \rightarrow \infty} e^{-ct/m} = gt(1) \quad [\text{because } -ct/m \rightarrow 0 \text{ as } m \rightarrow \infty] = gt.$$

The speed of a very heavy falling object is approximately proportional to the elapsed time—it doesn't depend on the mass.