Mathematics 1a, Section 3.1 Solutions

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1. a. e is the number such that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

b.

x	$(2.7^x - 1)/x$
-0.001	0.9928
-0.0001	0.9932
0.001	0.9937
0.0001	0.9933

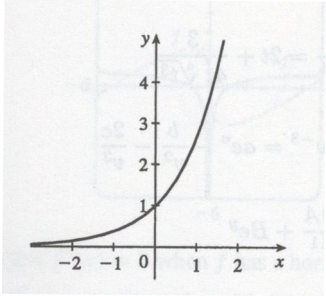
x	$(2.8^x - 1)/x$
-0.001	1.0291
-0.0001	1.0296
0.001	1.0301
0.0001	1.0297

From the tables (to two decimal places),

$$\lim_{h \to 0} \frac{2.7^h - 1}{h} = 0.99$$
$$\lim_{h \to 0} \frac{2.8^h - 1}{h} = 1.03$$

Since $0.99 < 1 < 1.03$ and $2.7 < e < 2.8$.

2. a.



The function value at $x = 0$ is 1 and the slope at $x = 0$ is 1.

b. $f(x) = e^x$ is an exponential function and $g(x) = x^e$ is a power function. $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(x^e) = ex^{e-1}$.

c. $f(x) = e^x$ grows more rapidly than $g(x) = x^e$ when x is large.

6.

$$g(x) = 5x^8 - 2x^5 + 6$$
$$g'(x) = 5(8x^{8-1}) - 2(5x^{5-1}) + 0 = 40x^7 - 10x^4$$

19.

$$v = t^2 - \frac{1}{\sqrt[4]{t^3}} = t^2 - t^{\frac{-3}{4}}$$
$$v' = 2t - \left(-\frac{3}{4}\right) t^{-7/4} = 2t + \frac{3}{4t\sqrt[4]{t^3}}$$

20.

$$y = ae^v + \frac{b}{v} + \frac{c}{v^2} = ae^v + bv^{-1} + cv^{-2}$$
$$y' = ae^v - bv^{-2} - 2cv^{-3} = ae^v - \frac{b}{v^2} - 2\frac{c}{v^3}$$

38.

$$G(r) = \sqrt{r} + \sqrt[3]{r}$$
$$G'(r) = \frac{1}{2}t^{-1/2} + \frac{1}{3}t^{-2/3}$$
$$G''(r) = -\frac{1}{4}r^{-3/2} - \frac{2}{9}r^{-5/3}$$

50. If $y = x^2 + x$ then $y' = 2x + 1$. If the point at which a tangent meets the parabola is $(a, a^2 + a)$, then the slope of the tangent is $2a + 1$. But since it passes through $(2, -3)$, the slope must also be

$$\frac{\Delta y}{\Delta x} = \frac{a^2 + a + 3}{a - 2}$$

Therefore, $2a + 1 = \frac{a^2+a+3}{a-2}$. Solving this equation for a we get $a^2 + a + 3 = 2a^2 - 3a - 2 \Rightarrow a^2 - 4a - 5 = (a - 5)(a + 1) = 0 \Rightarrow a = 5$ or $a = -1$. If $a = -1$ the point is $(-1, 0)$ and the slope is -1, so the equation is $y = -x - 1$. If $a = 5$, the point is $(5, 30)$ and the slope is 11,

so the equation is $y = 11x - 25$.

64. In order for the two tangents to intersect on the y-axis, the points of tangency must be at equal distances from the y-axis, since the parabola $y = x^2$ is symmetric about the y-axis. Say the points of tangency are (a, a^2) and $(-a, a^2)$, for some $a > 0$. Then since the derivative of $y = x^2$ is $y' = 2x$, the left-hand tangent has slope $-2a$ and equation $y - a^2 = -2a(x + a)$, or $y = -2ax - a^2$, and similarly the right-hand tangent line has equation $y - a^2 = 2a(x - a)$, or $y = 2ax - a^2$. So the two lines intersect at $(0, -a^2)$. Now if the lines are perpendicular, then the product of their slopes is -1, so $(-2a)(2a) = -1 \Rightarrow a^2 = \frac{1}{4} \Rightarrow a = \frac{1}{2}$. So the lines intersect at $(0, -\frac{1}{4})$.