Name: ________________________ ID#: ________________

# Solutions to Midterm III

Math 1a
Introduction to Calculus

May 6, 2005

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

Problems are numbered with arabic numerals (1, 2, 3, ...) and may stretch across several pages. Parts of problems are enumerated with either letters ((a), (b), (c), ...) or small roman numerals ((i), (ii), (iii), ...).

**This is a non-calculator exam.**

> *Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.*
>
> —*Handbook for Students*

**1.** (20 Points) *Compute the following integrals. For definite integrals, your answer should be a number. For indefinite integrals, your answer should be the most general antiderivative as a function of* $x$*.*

*(i)* $\displaystyle\int x^{3/2}\,dx.$

*Solution.* By the power rule, we have

$$\int x^{3/2}\,dx = \frac{x3/2+1}{3/2+1} = \frac{2}{5}x^{5/2} + C.$$

□

*(ii)* $\displaystyle\int_{-3}^{5}\left(5-\frac{x}{2}\right)\,dx$

*Solution.* Using the second Fundamental Theorem of Calculus, we have

$$\begin{aligned}\int_{-3}^{5}\left(5-\frac{x}{2}\right)\,dx &= \left[5x-\frac{x^2}{4}\right]_{-3}^{5}\\ &= \left(5(5)-\frac{5^2}{4}\right)-\left(5(-3)-\frac{(-3)^2}{4}\right)\\ &= 25+15-\frac{25}{4}+\frac{9}{4}\\ &= 40-\frac{16}{4} = 36.\end{aligned}$$

□

*(iii)* $\int x^3(x^4-1)^2\,dx$

*Solution.* There are two ways to do this problem:

- *The Hard Way.* Expand the integrand and integrate term-by-term:

$$\begin{aligned}\int x^3(x^4-1)^2\,dx &= \int \left(x^3\left(x^8-2x^4+1\right)\right)\,dx \\ &= \int \left(x^{11}-2x^7+x^3\right)\,dx \\ &= \frac{1}{12}x^{12}-\frac{1}{4}x^8+\frac{1}{4}x^4+C.\end{aligned}$$

- *The Easy Way.* Use the substitution $u = x^4-1$, so that $du = 4x^3\,dx$ and

$$\int x^3(x^4-1)^2\,dx = \frac{1}{4}\int u^2\,du = \frac{1}{12}u^3 = \frac{1}{12}(x^4-1)^3+C.$$

Notice that

$$\frac{1}{12}(x^4-1)^3 = \frac{1}{12}x^{12}-\frac{1}{4}x^8+\frac{1}{4}x^4-\frac{1}{12},$$

so these two answers differ by a constant. □

*(iv)* $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\,dx.$

*Solution.* Let $u=\cos x$, so that $du = -\sin x\,dx$. Then also $u(2\pi)=1$ and $u(3\pi)=-1$. So

$$\begin{aligned}\int_{2\pi}^{3\pi} 3\cos^2 x\sin x\,dx &= -\int_1^{-1} 3u^2\,du = \int_{-1}^{1} 3u^2\,du \\ &= u^3\big|_{-1}^{1} = 2.\end{aligned}$$

□



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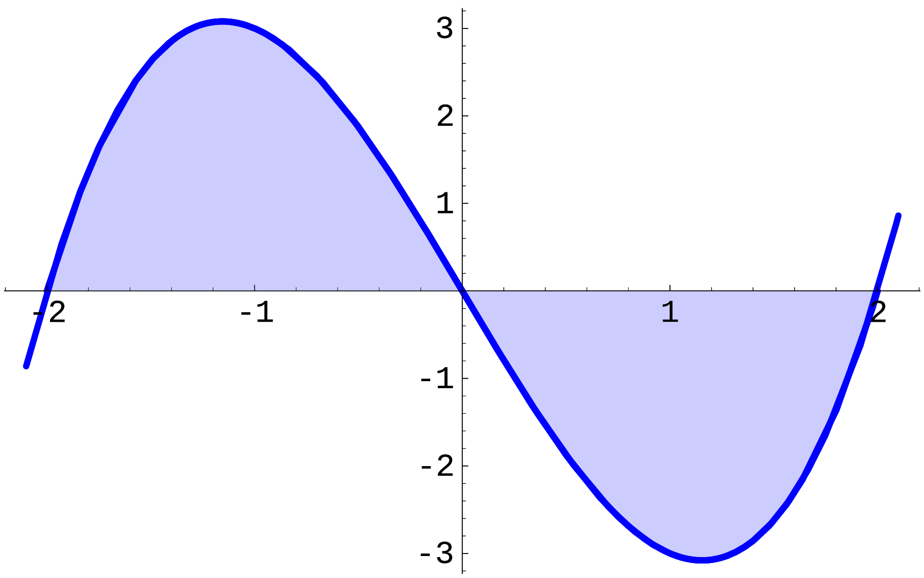
**2.** (15 Points) *Find the total area of the region bounded by the curve* $y = x^3 - 4x$ *and the* $x$*-axis between* $-2$ *and* $2$*.*




*Solution.* By factoring the integrand

$$x^3 - 4x = x(x^2 - 4) = x(x-2)(x+2),$$

we have that the function is positive on (-2,0) and negative on (0,2). So we have

$$\begin{aligned} A &= \int_{-2}^{2} \left| x^3 - 4x \right| \, dx \\ &= \int_{-2}^{0} (x^3 - 4x) \, dx - \int_{0}^{2} (x^3 - 4x) \, dx \\ &= 2 \int_{0}^{2} (4x - x^3) \, dx. \end{aligned}$$

(The last step is a trick to simplify. It's not necessary, but does it make sense?). Continuing,

$$\begin{aligned} A &= 2 \left[ 2x^2 - \frac{x^4}{4} \right]_0^2 \\ &= 2 \left[ 2(2)^2 - \frac{2^4}{4} \right] = 2(4) = 8. \end{aligned}$$

$\square$

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**3.** (15 Points) *The police observe that the skid marks made by a stopping car are 250 feet long. Assuming the car decelerated at a constant rate of* $20\frac{feet}{sec^2}$*, skidding all the way, how fast was the car traveling when the brakes were initially applied?*

*Solution.* Let $s$ be the position function of the car versus time, $v = s'$ the velocity, and $a = s''$ the acceleration. Moving backwards from the statement that $a = -20$, we have

$$v(t) = v_0 - 20t;$$
$$s(t) = s_0 + v_0 t - 10t^2.$$

The car is at $s_0$ and traveling at a velocity of $v_0$ when hitting the brakes. Let us suppose that at time $t$ the car is stopped ($v(t) = 0$); we know that by this time the car has traveled 250 feet. So $s(t) - s_0 = 250$. We can assume $s_0 = 0$ (we will just measure from the start of the skid) and so we have a pair of equations:

$$v_0 - 20t = 0;$$
$$v_0 t - 10t^2 = 250.$$

We need to find $v_0$. Solving the first for $t$ gives $t = \frac{v_0}{20}$; plugging this into the second gives

$$v_0\left(\frac{v_0}{20}\right) - 10\left(\frac{v_0}{20}\right)^2 = 250$$
$$\frac{v_0^2}{40} = 250$$
$$v_0^2 = 10,000$$
$$v_0 = 100\frac{\text{ft}}{\text{sec}}.$$

This is about 68 miles per hour. □



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**4.** (10 Points) *Let $f$ be the function*

$$f(x) = x \int_0^x \frac{\sin t}{t}\, dt.$$

*(The integrand is continuous at 0 since*

$$\lim_{t \to 0} \frac{\sin t}{t} = 1;$$

*you might need that fact later).*

*(a) Show $f'(0) = 0$.*

*(b) It's not immediately clear what $f''(0)$ is. Find the limit:*

$$f''(0) = \lim_{x \to 0} \frac{f'(x)}{x} =?$$

*Solution.* We have

$$\begin{aligned} f'(x) &= \mathrm{Si}(x) + x\,\mathrm{Si}'(x) \\ &= \mathrm{Si}(x) + x \cdot \frac{\sin(x)}{x} \\ &= \mathrm{Si}(x) + \sin x. \end{aligned}$$

So $f'(0) = \mathrm{Si}(0) + \sin(0) = 0$. Hence 0 is a critical point of $f$. Also,

$$\begin{aligned} f''(x) &= \mathrm{Si}'(x) + \sin'(x) \\ &= \frac{\sin x}{x} + \cos x. \end{aligned}$$

So $f''(0) = 1 + 1 = 2$. Here we have used the limit fact stated above. □

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