Name: ______________________________ ID#: ____________________

Solutions to Midterm II

Math 1a
Introduction to Calculus

April 18, 2005

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

> *Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.*
>
> —*Handbook for Students*

1. (20 Points) *Find the following derivatives:*

(i) $\dfrac{d}{dt}\sin^3\left(\dfrac{2}{t}\right)$

Solution. We have

$$\begin{aligned}\frac{d}{dt}\sin^3\left(\frac{2}{t}\right) &= 3\sin^2\left(\frac{2}{t}\right)\frac{d}{dt}\sin\left(\frac{2}{t}\right)\\ &= 3\sin^2\left(\frac{2}{t}\right)\cos\left(\frac{2}{t}\right)\frac{d}{dt}\frac{2}{t}\\ &= 3\sin^2\left(\frac{2}{t}\right)\cos\left(\frac{2}{t}\right)\frac{-2}{t^2}\\ &= -\frac{6}{t^2}\sin^2\left(\frac{2}{t}\right)\cos\left(\frac{2}{t}\right)\end{aligned}$$

□

(ii) $\dfrac{d}{d\theta}\ln(\sec^2\theta)$

Solution. There are a couple of ways to do this. The brute-force way with the chain rule:

$$\frac{d}{d\theta}\ln(\sec^2\theta) = \frac{1}{\sec^2\theta}(2\sec\theta)(\sec\theta\tan\theta) = 2\tan\theta.$$

On the other hand we can use the properties of logarithms to get

$$\frac{d}{d\theta}\ln(\sec^2\theta) = \frac{d}{d\theta}(-2\ln\cos\theta) = -2\frac{1}{\cos\theta}(-\sin\theta) = 2\tan\theta.$$

□

(iii) $\frac{dy}{dx}$, *where* $xy + 2x + 3y = 1$ *(your answer should involve* x *and* y*).*

Solution. Taking the derivative of both sides with respect to x we have

$$y + x\frac{dy}{dx} + 2 + 3\frac{dy}{dx} = 0$$
$$\implies \frac{dy}{dx} = -\frac{y+x}{x+3}.$$

□

(iv) $\frac{d}{dx}2(x^2+1)^{x/2}$

Solution. Let y be the expression above, so

$$\begin{aligned}
\ln y &= \left(\frac{x}{2}\right)\ln(2x^2+2) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{2}\ln(2x^2+2) + \left(\frac{x}{2}\right)\frac{4x}{2x^2+2} \\
\implies \frac{dy}{dx} &= \left\{\frac{1}{2}\ln(2x^2+2) + \left(\frac{x}{2}\right)\frac{4x}{2x^2+2}\right\}2(x^2+1)^{x/2} \\
&= \left\{\ln(2x^2+2) + \frac{2x^2}{x^2+1}\right\}(x^2+1)^{x/2}.
\end{aligned}$$

Incorrect use of the power rule, as in

$$\frac{d}{dx}2(x^2+1)^{x/2} \neq 2(x/2)(x^2+1)^{x/2-1}(2x)$$

resulted in a 1/5 score on this part. □

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2. (25 Points) *We are going to graph completely the function*

$$f(x) = \frac{3}{4}x^{1/3}(x-4).$$

(a) Find the domain of f.

Solution. The cube root is defined for all real numbers. Hence, the domain of this function is $\mathbb{R}$. □

(b) Find the places where f is positive, negative, or zero.

Solution. $f(x)$ is zero when one of its factors is zero, that is when $x = 0$ or $x = 4$. We can make a sign chart as follows:

	$x < 0$	$0 < x < 4$	$x > 4$
$x^{1/3}$	$-$	$+$	$+$
$x - 4$	$-$	$-$	$+$
f	$+$	$-$	$+$

□

Positive on : $(-\infty, 0)$, $(4, \infty)$

Negative on : $(0, 4)$

Zero at: 0, 4

(c) Find all horizontal and vertical asymptotes (if any) of the graph of f.

Solution. There are no vertical asymptotes since f is defined for all x. We can check that

$$\lim_{x\to\infty} f(x) = \infty;$$
$$\lim_{x\to-\infty} f(x) = -\infty.$$

So there are no horizontal asymptotes, either. □

(d) *The derivative of f is*

$$f'(x) = x^{-2/3}(x-1).$$

Find the intervals of increase or decrease.

Solution. I gave this derivative so you didn't have to calculate it. All we need is the critical points and the sign chart!

The function has critical points at 1 (where $f'(x) = 0$) and 0 (where f is not differentiable). We can make a sign chart:

	$x < 0$	$0 < x < 1$	$x > 1$
$x^{-2/3}$	$+$	$+$	$+$
$x - 1$	$-$	$-$	$+$
f'	$-$	$-$	$+$
f	$\searrow$	$\searrow$	$\nearrow$

f is decreasing on the whole interval $(-\infty, 1)$, even though f is not differentiable at 0. □

Increasing on: $(1, \infty)$

Decreasing on: $(-\infty, 0)$

(e) *Find any local maxima or minima.*

Solution. Apparently there is a local minimum at $(1, -9/4)$, where f changes from decreasing to increasing. □

(f) *The second derivative of f is*

$$f''(x) = \frac{x+2}{3x^{5/3}}.$$

Find the intervals of concavity.

Solution. The points of interest are $x = -2$ (where $f''(x) = 0$) and $x = 0$ (where f' is not differentiable). The sign chart is

	$x < -2$	$-2 < x < 0$	$x > 0$
$x^{-5/3}$	$-$	$-$	$+$
$x+2$	$-$	$+$	$+$
f''	$++$	$--$	$++$
f	$\smile$	$\frown$	$\smile$

□

Concave up on: $(-\infty, -2)$, $(0, \infty)$

Concave down on: $(-2, 0)$

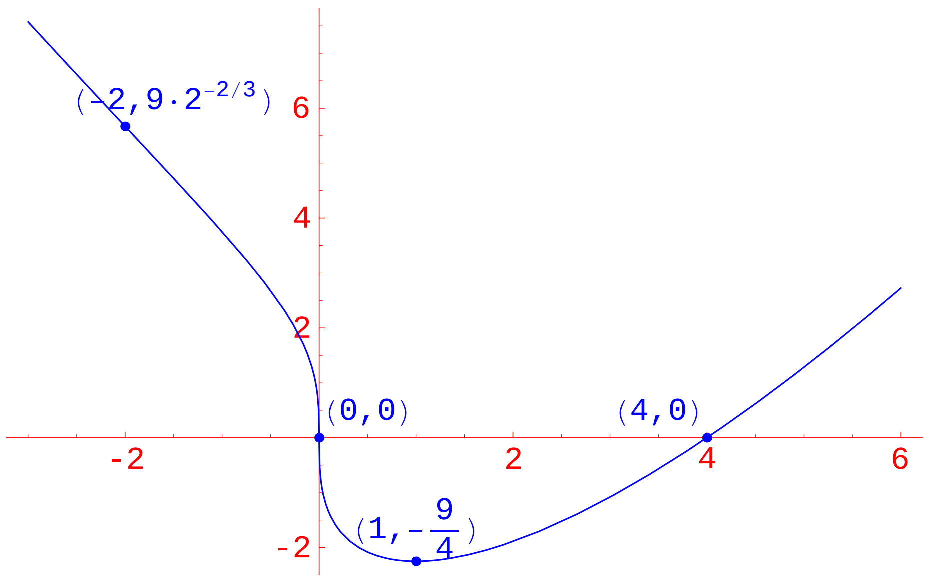
(g) *Find any inflection point(s).*

Solution. There are two inflection points: $(-2, 9 \cdot 2^{-2/3})$ and $(0, 0)$. Even though f is not twice-differentiable at 0, f'' changes sign on either side of 0, so it is still an inflection point. □

(h) Sketch the graph of f. Label all significant data–zeroes, asymptotes, local extrema, inflection points

Solution. We put all the information from (a)–(g) into a chart:

	$x < -2$	$-2 < x < 0$	$0 < x < 1$	$1 < x < 4$	$x > 4$
sign	+	+	−	−	+
monotonicity	↘	↘	↘	↗	↗
concavity	⌣	⌢	⌣	⌣	⌣
shape	◟	◝	◟	◞	◞



□

(i) Find the global minimum and maximum, if they exist.

Solution. There is no global maximum since the function increases without bound at $\pm\infty$. The local minimum $(1, -9/4)$ is the global minimum. □

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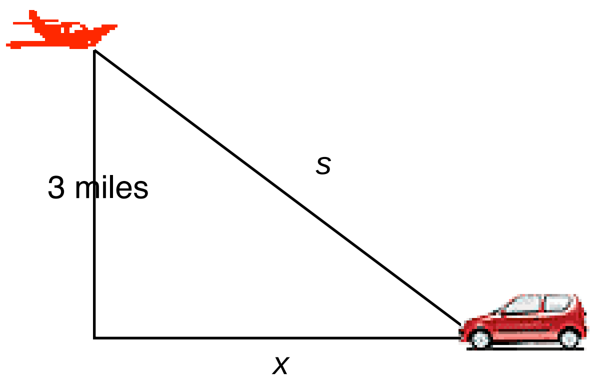
3. (15 Points) *A highway patrol plane flies 3 miles above a level, straight road at a steady 120 miles per hour. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 miles, this distance is decreasing at the rate of 160 miles per hour. Find the car's speed along the highway.*

Solution. Let x be the horizontal distance (along the road) between the car and the plane, and s the line-of-sight distance from the plane to the car, both measured in miles.



We know that

$$\left.\frac{ds}{dt}\right|_{h=5} = -160.$$

Once we know $\left.\frac{dx}{dt}\right|$, we will know the car's speed.

By the Pythagorean theorem,

$$x^2 + 3^2 = s^2,$$

so

$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}.$$

When $h = 5$, we have $x = 4$ miles and $\frac{ds}{dt} = -160\,\frac{\text{mi}}{\text{hour}}$. Now at the given time we have $s = 5$ miles and $\frac{ds}{dt} = -160\,\frac{\text{mi}}{\text{hour}}$. Thus

$$\left.\frac{dx}{dt}\right|_{h=5} = \frac{5}{4}(-160) = -200\,\frac{\text{mi}}{\text{hour}}.$$

This is the speed at which the car and plane (more precisely, the plane's shadow on the ground) are approaching each other Since the plane is moving at 120 miles/hour, and the car is moving in the opposite direction, it must be moving at 80 miles/hour. □

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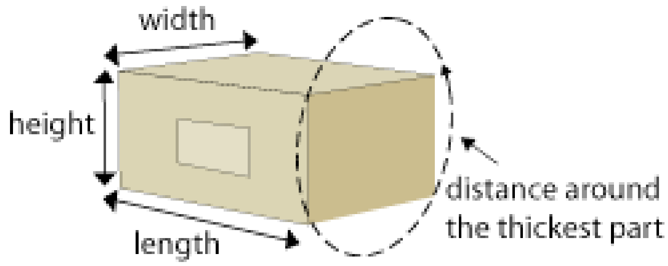
4. (15 Points) *the U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 108 inches.*



What dimensions will give a box with a square end (width and height are the same) the largest possible volume? Make sure you check that you have found the largest possible volume, not the smallest!

Solution. The function we are trying to maximize is $V = \ell wh$. But if the end is square, we have $w - h$, so instead we are trying to maximize $V = \ell w^2$. The length + girth of this box is $\ell + 4w$, which we constrain to be 108 inches.[1]

To solve for length in terms of width, we have

$$\ell + 4w = 108 \implies \ell = 108 - 4w,$$

so our new V is

$$V(w) = (108 - 4w)w^2 = 108w^2 - 4w^3.$$

The domain that we are concerned about is that which would keep the volume positive, i.e., $0 \le w \le 36$.[2] To maximize this,

$$\frac{dV}{dw} = 216w - 12w^2 = 12w(18 - w),$$

which is zero when $w = 18$. At this width the length is 36, so the volume is $(36)(18)(18) = 11,664\text{in}^3$. This is maximal since at the endpoints, $V = 0$. The other dimensions are $\ell = 36$ and $h = 18$. □

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[1] I apologize for the possible ambiguity of the diagram. Many thought "girth" was the circumference of the indicated circle. In this case you have a constraint which would look like

$$\ell + h\pi\sqrt{2} = 108.$$

I only took off one point for this assumption if the rest of the problem was finished. Incidentally, I got the picture from a USPS web site; they ought to know they're confusing the public!

[2] This is not exactly the whole story. If the length becomes less than the width, the girth becomes $2\ell + 2w$ instead of $4w$. Basically, girth is the perimeter of the smallest cross-section of the box, or the length of the shortest rubber band that fits all around the box. This makes the length + girth of the box piecewise defined and not always differentiable. But this problem has become complicated enough; I accepted any plausible argument that this point was the global maximum.