Name: _____ ID#: _____

Final Exam

Math 1a
Introduction to Calculus

21 May 2004

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

> *Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.*
>
> —*Handbook for Students*

Problem Number	Possible Points	Points Earned
1	15	
2	20	
3	15	
4	15	
5	20	
6	15	
7	20	
8	15	
9	15	
Total	150	

1 1

1. (15 Points) Find the following limits.

(i) $\lim_{x \to \infty} \dfrac{1+x}{1-x}$

(ii) $\lim_{x \to 2} \dfrac{x^2-4}{x^2+4}$

1 **1**

(iii) $\lim_{x \to 0^+} (\cos x)^{1/x}$

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2 2

2. (20 Points) Find the derivatives of the following functions.

(i) $f(x) = x^3$

(ii) $f(x) = 3^x$.

2 2

(iii) $f(x) = \sin(x^2)$

(iv) $f(x) = (\sin x)^2$

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3 3

3. (15 Points) A fish tank is to be designed with square base and rectangular sides (and no top). The material for the base costs five times as much as the glass for the sides. What are the dimensions of the tank which has volume 20ft^3 and costs the least to build?

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4 4

4. (15 Points) 1 is a pretty lousy approximation to $\sqrt[3]{2}$. Using Newton's method, find a rational number (whole number or fraction) which is closer, and another one which is still closer. Simplify your answer to a reasonable form.

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5 5

5. (20 Points) Jake and Elwood are organizing a concert to benefit the orphanage in which they grew up. They have determined through market research that if they set the price of tickets to be \$1, they will sell 7500 tickets, and if they set the price of tickets to be \$3, they will sell 2500 tickets.

(a) Assuming that the demand function (number of tickets sold in terms of price charged) $x(p)$ is linear, show that

$$x(p) = 10,000 - 2500p.$$

5 **5**

(b) Recalling that

$$\text{revenue} = \text{quantity sold} \times \text{price charged},$$

what ticket price will maximize their revenue?

☐ Check the box if you know what movie this problem comes from.

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6 6

6. (15 Points) Show that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2+i^2} = \frac{\pi}{4}.$$

Hint. $\dfrac{n}{n^2+i^2} = \dfrac{1}{1+\left(\frac{i}{n}\right)^2}\dfrac{1}{n}$.

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7 7

7. (20 Points) Compute the following integrals. For definite integrals, your answer should be a number. For indefinite integrals, your answer should be the most general antiderivative as a function of x.

(i) $\displaystyle\int_1^2 (x^3 + 4)\, dx$

(ii) $\displaystyle\int x e^{-x^2}\, dx$

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(iii) $\displaystyle\int_1^2 \frac{(\ln x)^6}{x}\,dx$

(iv) $\displaystyle\int \sin^3 x\,dx$

Hint. $\sin^2 x = 1 - \cos^2 x$.

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8 8

8. (15 Points)

(a) (5 points) Draw the region between the graph of $y = \sin 3x$, the x-axis, and the vertical lines $x = 0$ and $x = \pi/3$.

(b) (10 points) Find its area.

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9 9

9. (15 Points) Let k be a constant and

$$F(y) = \int_0^y \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}.$$

(a) Find $F'(y)$. (It will have a k in it, but not a t.)

Define a new function $a(x) = F^{-1}(x)$, called the *amplitude* of x, so that

$$F(a(x)) = x \qquad (*)$$

is true for all x. Also define

$$\begin{aligned} s(x) &= \sin(a(x)); \\ c(x) &= \cos(a(x)); \\ d(x) &= \sqrt{1 - k^2 \sin^2(a(x))} \end{aligned}$$

(these are called the *Jacobi elliptic functions*).

(b) Use (a) and $(*)$ to show

$$\frac{d}{dx} a(x) = d(x).$$

9 9

(c) Show

$$\frac{d}{dx}d(x) = -k^2 s(x)c(x).$$

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