3.3

**1.** (a) $s = f(t) = t^3 - 12t^2 + 36t \Rightarrow v(t) = f'(t) = 3t^2 - 24t + 36$

(b) $v(3) = 27 - 72 + 36 = -9$ m/s

(c) The particle is at rest when $v(t) = 0$. $3t^2 - 24t + 36 = 0 \Rightarrow 3(t-2)(t-6) = 0 \Rightarrow t = 2, 6$.

(d) The particle is moving in the positive direction when $v(t) > 0$. $3(t-2)(t-6) > 0 \Leftrightarrow 0 \le t < 2$ or $t > 6$.

(e) Since the particle is moving forward and backward, we need to calculate the distance traveled in the intervals $[0, 2]$, $[2, 6]$, and $[6, 8]$ separately.

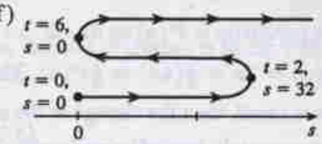
$|f(2) - f(0)| = |32 - 0| = 32$.

$|f(6) - f(2)| = |0 - 32| = 32$.

$|f(8) - f(6)| = |32 - 0| = 32$.

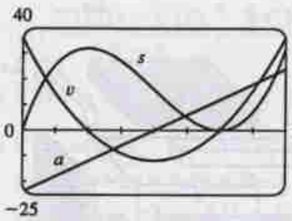
The total distance is $32 + 32 + 32 = 96$ m.

(f)




(g) $a(t) = v'(t) = 6t - 24$. $a(3) = 6(3) - 24 = -6$ (m/s)/s or m/s$^2$.

(h)




(i) The particle is speeding up when $v$ and $a$ have the same sign. This occurs when $2 < t < 4$ and when $t > 6$. It is slowing down when $v$ and $a$ have opposite signs; that is, when $0 \le t < 2$ and when $4 < t < 6$.

**2.** (a) $x(t) = \dfrac{t}{1+t^2} \Rightarrow v(t) = x'(t) = \dfrac{(1+t^2)(1) - t(2t)}{(1+t^2)^2} = \dfrac{1-t^2}{(1+t^2)^2}$

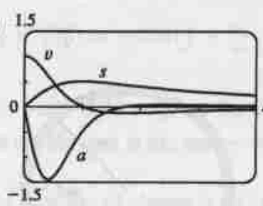
(b) Right: $v(t) > 0 \Rightarrow 1 - t^2 > 0 \Rightarrow t^2 < 1 \Rightarrow |t| < 1 \Rightarrow 0 \le t < 1$

Left: $v(t) < 0 \Rightarrow 1 - t^2 < 0 \Rightarrow t > 1$

(c) $|x(1) - x(0)| + |x(4) - x(1)| = \left|\frac{1}{2} - 0\right| + \left|\frac{4}{17} - \frac{1}{2}\right| = \frac{1}{2} + \frac{9}{34} = \frac{13}{17}$

(d) $a(t) = v'(t) = \dfrac{2t(t^2-3)}{(1+t^2)^3}$. $a(t) = 0 \Rightarrow 2t(t^2 - 3) = 0 \Rightarrow t = 0$ or $\sqrt{3}$

(e)




(f) $v$ and $a$ have the same sign and the particle is speeding up when $1 < t < \sqrt{3}$. The particle is slowing down and $v$ and $a$ have opposite signs when $0 < t < 1$ and when $t > \sqrt{3}$.

**4.** (a) At maximum height the velocity of the ball is 0 ft/s. $v(t) = s'(t) = 80 - 32t = 0 \Leftrightarrow 32t = 80 \Leftrightarrow t = \frac{5}{2}$. So the maximum height is $s\left(\frac{5}{2}\right) = 80\left(\frac{5}{2}\right) - 16\left(\frac{5}{2}\right)^2 = 200 - 100 = 100$ ft.

(b) $s(t) = 80t - 16t^2 = 96 \Leftrightarrow 16t^2 - 80t + 96 = 0 \Leftrightarrow 16(t^2 - 5t + 6) = 0 \Leftrightarrow 16(t-3)(t-2) = 0$. So the ball has a height of 96 ft on the way up at $t = 2$ and on the way down at $t = 3$. At these times the velocities are $v(2) = 80 - 32(2) = 16$ ft/s and $v(3) = 80 - 32(3) = -16$ ft/s, respectively.

**14.** (a) $F = \dfrac{GmM}{r^2} = (GmM)r^{-2} \;\Rightarrow\; \dfrac{dF}{dr} = -2(GmM)r^{-3} = -\dfrac{2GmM}{r^3}$, which is the rate of change of the force with respect to the distance between the bodies. The minus sign indicates that as the distance $r$ between the bodies increases, the magnitude of the force $F$ exerted by the body of mass $m$ on the body of mass $M$ is decreasing.

(b) Given $F'(20{,}000) = -2$, find $F'(10{,}000)$. $-2 = -\dfrac{2GmM}{20{,}000^3} \;\Rightarrow\; GmM = 20{,}000^3$.

$$F'(10{,}000) = -\frac{2(20{,}000^3)}{10{,}000^3} = -2 \cdot 2^3 = -16 \text{ N/km}$$

**20.** (a) After an hour the population is $n(1) = 3 \cdot 500$; after two hours it is $n(2) = 3(3 \cdot 500) = 3^2 \cdot 500$; after three hours, $n(3) = 3(3^2 \cdot 500) = 3^3 \cdot 500$; after four hours, $n(4) = 3^4 \cdot 500$. From this pattern, we see that the population after $t$ hours is $n(t) = 3^t \cdot 500 = 500 \cdot 3^t$.

(b) From (5) in Section 3.1, we have $\dfrac{d}{dx}(3^x) \approx (1.10)3^x$. Thus, for $n(t) = 500 \cdot 3^t$,

$$\frac{dn}{dt} = 500\,\frac{d}{dt}(3^t) \approx 500(1.10)3^t \;\Rightarrow\; \left.\frac{dn}{dt}\right|_{t=6} \approx 500(1.10)3^6 \approx 400{,}950 \text{ bacteria/hour.}$$

**22.** (a) (i) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-1} \;\Rightarrow\; \dfrac{df}{dL} = -\left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-2} = -\dfrac{1}{2L^2}\sqrt{\dfrac{T}{\rho}}$

(ii) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{1/2} \;\Rightarrow\; \dfrac{df}{dT} = \dfrac{1}{2}\left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{-1/2} = \dfrac{1}{4L\sqrt{T\rho}}$

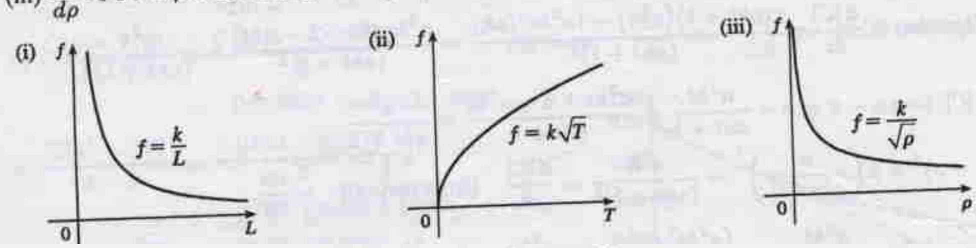
(iii) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-1/2} \;\Rightarrow\; \dfrac{df}{d\rho} = -\dfrac{1}{2}\left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-3/2} = -\dfrac{\sqrt{T}}{4L\rho^{3/2}}$

(b) *Note:* Illustrating tangent lines on the generic figures may help to explain the results.

(i) $\dfrac{df}{dL} < 0$ and $L$ is decreasing $\Rightarrow$ $f$ is increasing $\Rightarrow$ higher note

(ii) $\dfrac{df}{dT} > 0$ and $T$ is increasing $\Rightarrow$ $f$ is increasing $\Rightarrow$ higher note

(iii) $\dfrac{df}{d\rho} < 0$ and $\rho$ is increasing $\Rightarrow$ $f$ is decreasing $\Rightarrow$ lower note




**25.** (a) $A(x) = \dfrac{p(x)}{x} \;\Rightarrow\; A'(x) = \dfrac{xp'(x) - p(x)\cdot 1}{x^2} = \dfrac{xp'(x) - p(x)}{x^2}$. $A'(x) > 0 \;\Rightarrow\; A(x)$ is increasing; that is, the average productivity increases as the size of the work force increases.

(b) $p'(x)$ is greater than the average productivity $\Rightarrow$ $p'(x) > A(x) \;\Rightarrow\; p'(x) > \dfrac{p(x)}{x} \;\Rightarrow$

$$xp'(x) > p(x) \;\Rightarrow\; xp'(x) - p(x) > 0 \;\Rightarrow\; \frac{xp'(x) - p(x)}{x^2} > 0 \;\Rightarrow\; A'(x) > 0.$$

**29.** (a) If the populations are stable, then the growth rates are neither positive nor negative; that is, $\frac{dC}{dt} = 0$ and $\frac{dW}{dt} = 0$.

(b) "The caribou go extinct" means that the population is zero, or mathematically, $C = 0$.

(c) We have the equations $\frac{dC}{dt} = aC - bCW$ and $\frac{dW}{dt} = -cW + dCW$. Let $dC/dt = dW/dt = 0$, $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$ to obtain **(1)** $0.05C - 0.001CW = 0$ and **(2)** $-0.05W + 0.0001CW = 0$. Adding 10 times **(2)** to **(1)** eliminates the $CW$-terms and gives us $0.05C - 0.5W = 0 \quad \Rightarrow \quad C = 10W$. Substituting $C = 10W$ into **(1)** results in $0.05(10W) - 0.001(10W)W = 0 \quad \Leftrightarrow \quad 0.5W - 0.01W^2 = 0 \quad \Leftrightarrow \quad 50W - W^2 = 0 \quad \Leftrightarrow$ $W(50 - W) = 0 \quad \Leftrightarrow \quad W = 0$ or 50. Since $C = 10W$, $C = 0$ or 500. Thus, the population pairs $(C, W)$ that lead to stable populations are $(0, 0)$ and $(500, 50)$. So it is possible for the two species to live in harmony.