3.6 Implicit Differentiation

1. (a) $\frac{d}{dx}\left(xy + 2x + 3x^2\right) = \frac{d}{dx}(4) \Rightarrow (x \cdot y' + y \cdot 1) + 2 + 6x = 0 \Rightarrow xy' = -y - 2 - 6x \Rightarrow$

$y' = \frac{-y - 2 - 6x}{x}$ or $y' = -6 - \frac{y+2}{x}$.

(b) $xy + 2x + 3x^2 = 4 \Rightarrow xy = 4 - 2x - 3x^2 \Rightarrow y = \frac{4 - 2x - 3x^2}{x} = \frac{4}{x} - 2 - 3x$, so $y' = -\frac{4}{x^2} - 3$.

(c) From part (a), $y' = \frac{-y - 2 - 6x}{x} = \frac{-(4/x - 2 - 3x) - 2 - 6x}{x} = \frac{-4/x - 3x}{x} = -\frac{4}{x^2} - 3$.

4. $\frac{d}{dx}\left(x^2 - 2xy + y^3\right) = \frac{d}{dx}(c) \Rightarrow 2x - 2(xy' + y \cdot 1) + 3y^2y' = 0 \Rightarrow 2x - 2y = 2xy' - 3y^2y' \Rightarrow$

$2x - 2y = y'\left(2x - 3y^2\right) \Rightarrow y' = \frac{2x - 2y}{2x - 3y^2}$

13. $\frac{d}{dx}\left(\frac{x^2}{16} - \frac{y^2}{9}\right) = \frac{d}{dx}(1) \Rightarrow \frac{x}{8} - \frac{2yy'}{9} = 0 \Rightarrow y' = \frac{9x}{16y}$. When $x = -5$ and $y = \frac{9}{4}$ we have

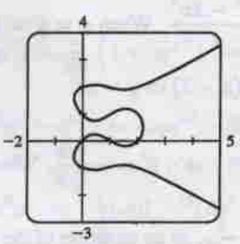
$y' = \frac{9(-5)}{16(9/4)} = -\frac{5}{4}$, so an equation of the tangent line is $y - \frac{9}{4} = -\frac{5}{4}(x + 5)$ or $y = -\frac{5}{4}x - 4$.

18. $x^2y^2 = (y+1)^2(4 - y^2) \Rightarrow 2x^2yy' + 2xy^2 = (y+1)^2(-2yy') + (4 - y^2) \cdot 2(y+1)y' \Rightarrow$

$xy^2 = \left[(y+1)(4-y^2) - y(y+1)^2 - x^2y\right]y' \Rightarrow y' = \frac{xy^2}{(y+1)(4-y^2) - y(y+1)^2 - x^2y} = 0$ when

$x = 0$. So an equation of the tangent line at $(0, -2)$ is $y + 2 = 0(x - 0)$ or $y = -2$.

21. (a)



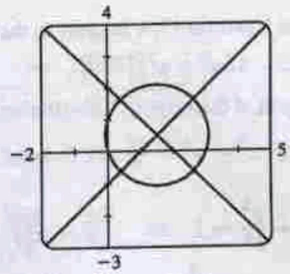
There are eight points with horizontal tangents: four at $x \approx 1.57735$ and four at $x \approx 0.42265$.

(b) $y' = \frac{3x^2 - 6x + 2}{2(2y^3 - 3y^2 - y + 1)} \Rightarrow y' = -1$ at $(0, 1)$ and $y' = \frac{1}{3}$ at $(0, 2)$.

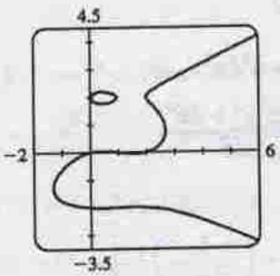
Equations of the tangent lines are $y = -x + 1$ and $y = \frac{1}{3}x + 2$.

(c) $y' = 0 \Rightarrow 3x^2 - 6x + 2 = 0 \Rightarrow x = 1 \pm \frac{1}{3}\sqrt{3}$

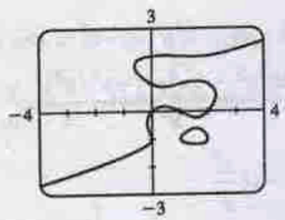
(d) By multiplying the right side of the equation by $x - 3$, we obtain the first graph.
By modifying the equation in other ways, we can generate the other graphs.



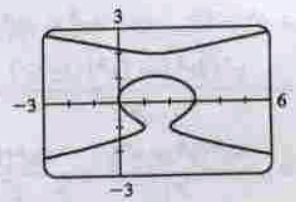
$y(y^2-1)(y-2)$
$= x(x-1)(x-2)(x-3)$



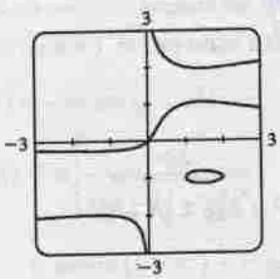
$y(y^2-4)(y-2)$
$= x(x-1)(x-2)$



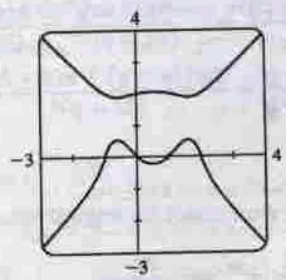
$y(y+1)(y^2-1)(y-2)$
$= x(x-1)(x-2)$



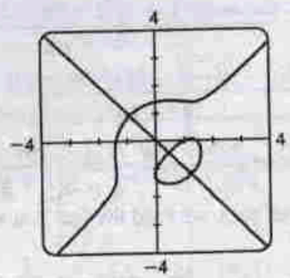
$(y+1)(y^2-1)(y-2)$
$= (x-1)(x-2)$



$x(y+1)(y^2-1)(y-2)$
$= y(x-1)(x-2)$



$y(y^2+1)(y-2)$
$= x(x^2-1)(x-2)$

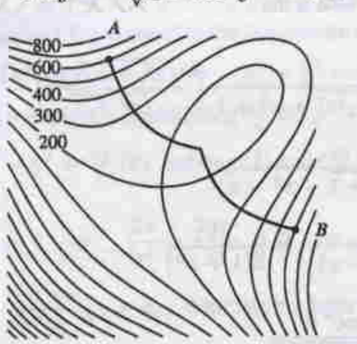


$y(y+1)(y^2-2)$
$= x(x-1)(x^2-2)$

34. Let $y = \cos^{-1} x$. Then $\cos y = x$ and $0 \le y \le \pi \;\Rightarrow\; \dfrac{d}{dx}(\cos y) = \dfrac{d}{dx}(x) \;\Rightarrow\; -\sin y \dfrac{dy}{dx} = 1 \;\Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\sin y} = -\dfrac{1}{\sqrt{1-\cos^2 y}} = -\dfrac{1}{\sqrt{1-x^2}}$ (Note that $\sin y \ge 0$ for $0 \le y \le \pi$.)

39.



43. $y = cx^2 \Rightarrow y' = 2cx$ and $x^2 + 2y^2 = k \Rightarrow 2x + 4yy' = 0 \Rightarrow$

$2yy' = -x \Rightarrow y' = -\dfrac{x}{2(y)} = -\dfrac{x}{2(cx^2)} = -\dfrac{1}{2cx}$, so the curves are orthogonal.