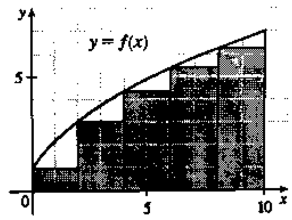
5.1

5 Integrals

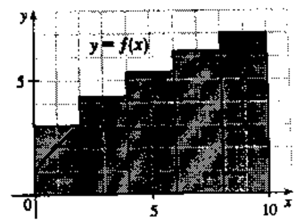
5.1 Areas and Distances

1. (a) Since f is *increasing*, we can obtain a *lower* estimate by using *left* endpoints. We are instructed to use five rectangles, so $n = 5$.



$$L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x \quad \left[\Delta x = \tfrac{b-a}{n} = \tfrac{10-0}{5} = 2\right]$$
$$= f(x_0)\cdot 2 + f(x_1)\cdot 2 + f(x_2)\cdot 2 + f(x_3)\cdot 2 + f(x_4)\cdot 2$$
$$= 2\,[f(0) + f(2) + f(4) + f(6) + f(8)]$$
$$\approx 2(1 + 3 + 4.3 + 5.4 + 6.3) = 2(20) = 40$$

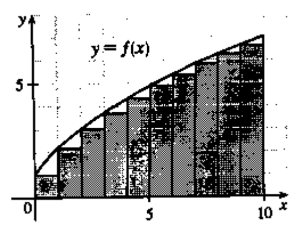
Since f is *increasing*, we can obtain an *upper* estimate by using *right* endpoints.



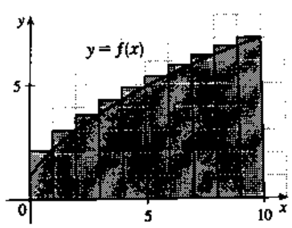
$$R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x$$
$$= 2\,[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$$
$$= 2\,[f(2) + f(4) + f(6) + f(8) + f(10)]$$
$$\approx 2(3 + 4.3 + 5.4 + 6.3 + 7) = 2(26) = 52$$

Comparing R_5 to L_5, we see that we have added the area of the rightmost rectangle, $f(10)\cdot 2$, to the sum and subtracted the area of the leftmost rectangle, $f(0)\cdot 2$, from the sum.

(b) $$L_{10} = \sum_{i=1}^{10} f(x_{i-1})\,\Delta x \quad \left[\Delta x = \tfrac{10-0}{10} = 1\right]$$
$$= 1\,[f(x_0) + f(x_1) + \cdots + f(x_9)]$$
$$= f(0) + f(1) + \cdots + f(9)$$
$$\approx 1 + 2.1 + 3 + 3.7 + 4.3 + 4.9 + 5.4 + 5.8 + 6.3 + 6.7$$
$$= 43.2$$

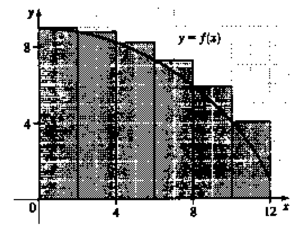


$$R_{10} = \sum_{i=1}^{10} f(x_i)\,\Delta x = f(1) + f(2) + \cdots + f(10)$$
$$= L_{10} + 1\cdot f(10) - 1\cdot f(0) \quad \begin{bmatrix}\text{add rightmost rectangle,}\\ \text{subtract leftmost}\end{bmatrix}$$
$$= 43.2 + 7 - 1 = 49.2$$

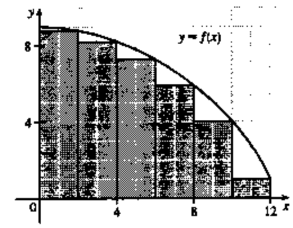


2. (a) (i) $L_6 = \sum_{i=1}^{6} f(x_{i-1})\Delta x \quad [\Delta x = \frac{12-0}{6} = 2]$

$= 2[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$
$= 2[f(0) + f(2) + f(4) + f(6) + f(8) + f(10)]$
$\approx 2(9 + 8.8 + 8.2 + 7.3 + 5.9 + 4.1)$
$= 2(43.3) = 86.6$

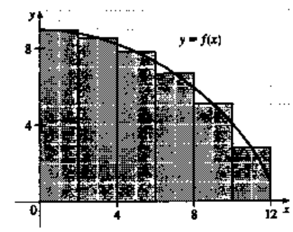


(ii) $R_6 = L_6 + 2 \cdot f(12) - 2 \cdot f(0)$
$\approx 86.6 + 2(1) - 2(9) = 70.6$



(iii) $M_6 = \sum_{i=1}^{6} f(x_i^*)\,\Delta x$

$= 2[f(1) + f(3) + f(5) + f(7) + f(9) + f(11)]$
$\approx 2(8.9 + 8.5 + 7.8 + 6.6 + 5.1 + 2.8)$
$= 2(39.7) = 79.4$



(b) Since f is *decreasing*, we obtain an *overestimate* by using *left* endpoints; that is, L_6.

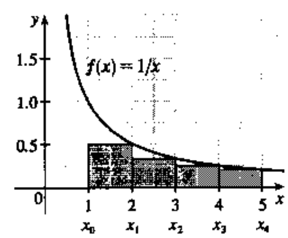
(c) Since f is *decreasing*, we obtain an *underestimate* by using *right* endpoints; that is, R_6.

(d) M_6 gives the best estimate, since the area of each rectangle appears to be closer to the true area than the overestimates and underestimates in L_6 and R_6.

3. (a) $R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x \quad [\Delta x = \frac{5-1}{4} = 1]$

$= f(x_1) \cdot 1 + f(x_2) \cdot 1 + f(x_3) \cdot 1 + f(x_4) \cdot 1$
$= f(2) + f(3) + f(4) + f(5)$
$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} = 1.28\overline{3}$

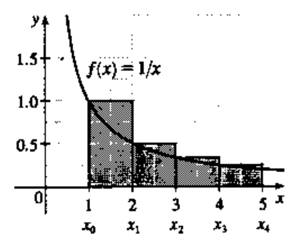
Since f is *decreasing* on $[1, 5]$, an *underestimate* is obtained by using the *right* endpoint approximation, R_4.



(b) $L_4 = \sum_{i=1}^{4} f(x_{i-1})\,\Delta x$

$= f(1) + f(2) + f(3) + f(4)$
$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} = 2.08\overline{3}$

L_4 is an overestimate. Alternatively, we could just add the area of the leftmost rectangle and subtract the area of the rightmost; that is, $L_4 = R_4 + f(1) \cdot 1 - f(5) \cdot 1$.



7. Here is one possible algorithm (ordered sequence of operations) for calculating the sums:

1 Let SUM = 0, X_MIN = 0, X_MAX = π, N = 10 (or 30 or 50, depending on which sum we are calculating), DELTA_X = (X_MAX − X_MIN)/N, and RIGHT_ENDPOINT = X_MIN + DELTA_X.

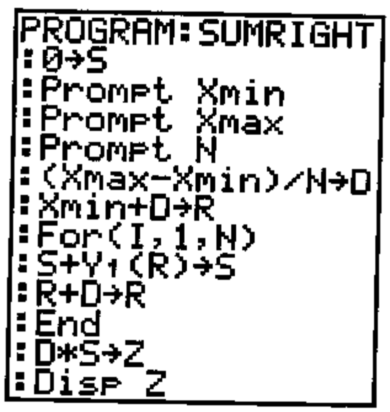
2 Repeat steps 2a, 2b in sequence until RIGHT_ENDPOINT > X_MAX.

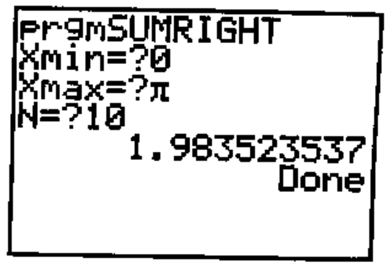
2a Add sin (RIGHT_ENDPOINT) to SUM.

2b Add DELTA_X to RIGHT_ENDPOINT.

At the end of this procedure, (DELTA_X) · (SUM) is equal to the answer we are looking for. We find that $R_{10} = \frac{\pi}{10}\sum_{i=1}^{10}\sin\left(\frac{i\pi}{10}\right) \approx 1.9835$, $R_{30} = \frac{\pi}{30}\sum_{i=1}^{30}\sin\left(\frac{i\pi}{30}\right) \approx 1.9982$, and $R_{50} = \frac{\pi}{50}\sum_{i=1}^{50}\sin\left(\frac{i\pi}{50}\right) \approx 1.9993$. It appears that the exact area is 2.

Shown below is program SUMRIGHT and its output from a TI-83 Plus calculator. To generalize the program, we have input (rather than assign) values for Xmin, Xmax, and N. Also, the function, $\sin x$, is assigned to Y_1, enabling us to evaluate any right sum merely by changing Y_1 and running the program.





11. Since v is an increasing function, L_6 will give us a lower estimate and R_6 will give us an upper estimate.

$L_6 = (0 \text{ ft/s})(0.5 \text{ s}) + (6.2)(0.5) + (10.8)(0.5) + (14.9)(0.5) + (18.1)(0.5) + (19.4)(0.5)$

$= 0.5(69.4) = 34.7 \text{ ft}$

$R_6 = 0.5(6.2 + 10.8 + 14.9 + 18.1 + 19.4 + 20.2) = 0.5(89.6) = 44.8 \text{ ft}$

12. We can find an upper estimate by using the final velocity for each time interval. Thus, the distance d traveled after 62 seconds can be approximated by

$d = \sum_{i=1}^{6} v(t_i)\Delta t_i = (185 \text{ ft/s})(10 \text{ s}) + 319 \cdot 5 + 447 \cdot 5 + 742 \cdot 12 + 1325 \cdot 27 + 1445 \cdot 3 = 54{,}694 \text{ ft}$

13. For a decreasing function, using left endpoints gives us an overestimate and using right endpoints results in an underestimate. We will use M_6 to get an estimate. $\Delta t = 1$, so

$$M_6 = 1[v(0.5) + v(1.5) + v(2.5) + v(3.5) + v(4.5) + v(5.5)]$$
$$\approx 55 + 40 + 28 + 18 + 10 + 4 = 155 \text{ ft}$$

For a very rough check on the above calculation, we can draw a line from $(0, 70)$ to $(6, 0)$ and calculate the area of the triangle: $\frac{1}{2}(70)(6) = 210$. This is clearly an overestimate, so our midpoint estimate of 155 is reasonable.

18. (a) $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and $x_i = 0 + i\,\Delta x = \frac{i}{n}$. $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \cdot \frac{1}{n}$.

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^3} \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3 = \lim_{n\to\infty} \frac{1}{n^4} \left[\frac{n(n+1)}{2}\right]^2 = \lim_{n\to\infty} \frac{(n+1)^2}{4n^2} = \frac{1}{4} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 = \frac{1}{4}$