1) We are given $xy^3 = 80$
and that $\frac{dy}{dt} = -0.8$ ft/sec when $y = 10$ ft

And we want $\frac{dx}{dt}$ when $y = 10$ ft

Use implicit differentiation:

$$xy^3 = 80$$

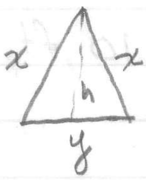
$$\Rightarrow \frac{dx}{dt} y^3 + 3y^2 \frac{dy}{dt} \cdot x = 0$$

$$\Rightarrow \frac{dx}{dt} = \frac{-3y^2 x \frac{dy}{dt}}{y^3}$$

when $y=10$, $x = \frac{80}{10^3} = .08$

$$\Rightarrow \frac{dx}{dt} = \frac{-3(.08)(-.8)}{10} = .019 \text{ ft/s}$$

2) $2x + y = p \quad \Rightarrow x = \frac{p-y}{2}$



Area $= A = \frac{1}{2} b h$

see that by Pythagoras

$$h = \sqrt{x^2 - \left(\frac{y}{2}\right)^2}$$

$$\Rightarrow \text{Area} = \frac{1}{2} y \sqrt{x^2 - \left(\frac{y}{2}\right)^2}$$

now substitute in for x:

$$A(y) = \frac{1}{2} y \sqrt{\left(\frac{p-y}{2}\right)^2 - \left(\frac{y}{2}\right)^2}$$
$$= \frac{1}{2} y \sqrt{\frac{p^2 - 2py + y^2 - y^2}{4}}$$

$$A(y) = \frac{1}{4} y \sqrt{p^2 - 2py}$$

$$A'(y) = \frac{1}{4}\sqrt{p^2 - 2py} + \frac{\frac{1}{4} y}{\sqrt{p^2 - 2py}} \cdot (-2p) \cdot \frac{1}{2}$$

now set this equal to zero and multiply by $4\sqrt{p^2 - 2py}$:

$$\Rightarrow p^2 - 2py - py = 0$$
$$\Rightarrow p - 3y = 0$$
$$\Rightarrow y = \frac{1}{3} p$$

Since the endpoints of the interval give $A = 0$ ft^2 (minima), we see that $y = x = \frac{1}{3} p$ gives the maximal area.

3) $f(x) = \frac{x^2 + 5x + 4}{x}$

First, examine limits: $\lim_{x \to \infty} f(x) = +\infty$, $\lim_{x \to -\infty} f(x) = -\infty$
also, we see that we have a vertical asymtote at $x = 0$, and $\lim_{x \to 0^+} = +\infty$, $\lim_{x \to 0^-} = -\infty$

Also, see that $f(x) = \frac{(x+4)(x+1)}{x}$

$\Rightarrow f(x) = 0$ for $x = -1, -4$, so we have the only roots at these points.

$$f'(x) = \frac{x(2x+5) - (x^2+5x+4)}{x^2} = \frac{2x^2 + 5x - x^2 - 5x - 4}{x^2}$$
$$= \frac{x^2 - 4}{x^2}$$

And we see $f'(x) = 0$ for $x = \pm 2$

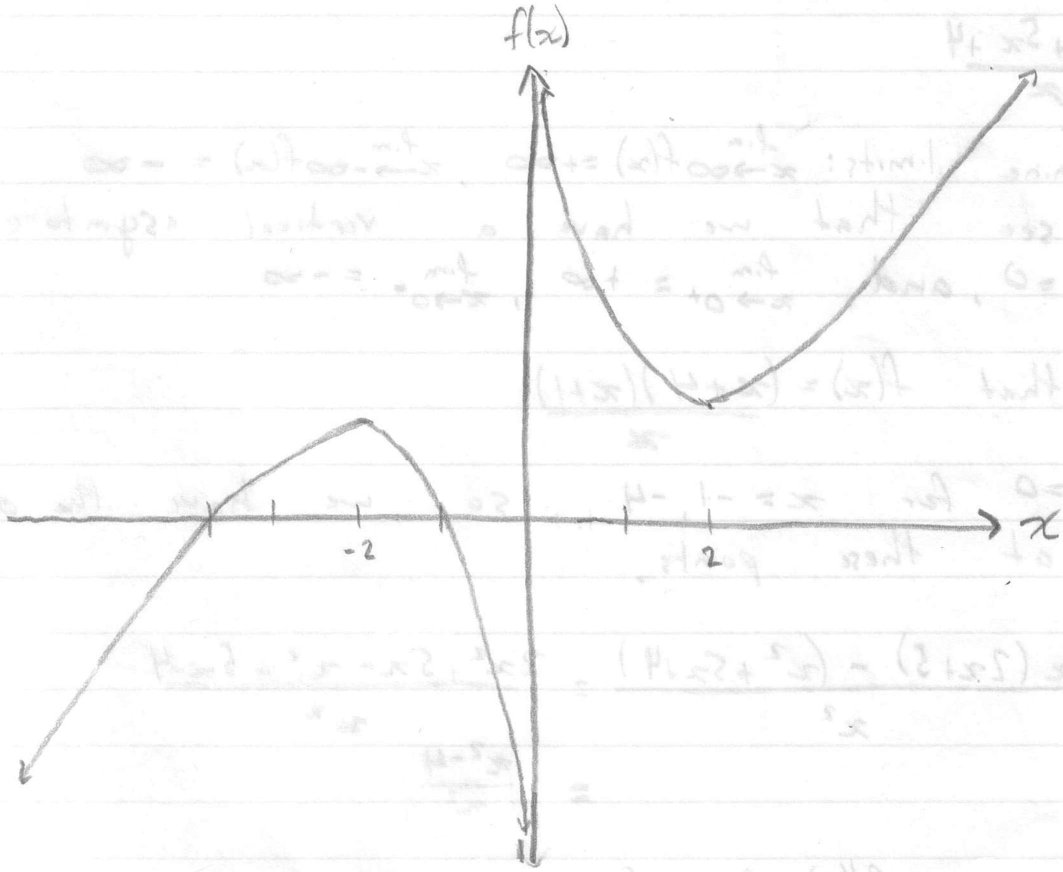
and since $f'(x) = \frac{x^2-4}{x^2} = 1 - \frac{4}{x^2}$

$$\Rightarrow f''(x) = -2 \cdot (-4) \cdot x^{-3} = \frac{8}{x^3}$$

and so $f''(x) < 0$ for $x < 0$
and $f''(x) > 0$ for $x > 0$

So f is concave down on $(-\infty, 0)$
and f is concave up on $(0, \infty)$

So now we can graph it!



f(x)
x
-2
2

(4) $f(x) = \dfrac{1}{1-e^x \sin 5x}$; $f'(x) = \dfrac{-(-e^x \cos 5x \cdot 5)}{(1-e^x \sin 5x)^2}$ $\dfrac{5e^x \cos 5x}{(1-e^x \sin 5x)}$

a) linear approximation around zero:

$$f(x) \approx f(0) + x \cdot f'(0)$$

$$f(0) = \frac{1}{1-1\cdot 0} = 1$$

$$f'(0) = \frac{5 \cdot 1 \cdot \cos 0}{(1-1\cdot \sin 0)^2} = 5$$

$\therefore$ $f(x) \approx 1+5x$

b) $f(0.03) \approx 1+5\times 0.03$

≈ 1.15

(5) a) $V = \int_0^t a\,dt + v_0$; $v_0 = 0$

$= \int_0^t \frac{t}{10}\,dt = \frac{t^2}{20}$ for $0 \le t \le 20$

$V_{20} = 20$

$V = \int_{20}^t a\,dt + V_{20}$; $V_{20} = 20$

$= \int_{20}^t 2\,dt + V_{20} = \left[2t\right]_{20}^t + 20$

$= 2t - 20$ for $20 \le t \le 40$

$\therefore V(30) = 60 - 20 = 40$ ft/sec²

b) s = distance

$s = \int_0^{30} v\,dt = \int_0^{20} \frac{t^2}{20}\,dt + \int_{20}^{30} 2t - 20\,dt$

$= \left[\frac{t^3}{60}\right]_0^{20} + \left[t^2 - 20t\right]_{20}^{30}$

$= \frac{8000}{60} + 900 - 600 + 400 - 400$

$= 433$ m.

⑥ a) $M = \int_0^{40} D(t)\, dt$

$$= \int_0^{20} D(t)\, dt + \int_{20}^{40} D(t)\, dt$$

$$= \int_0^{20} \frac{t}{10}\, dt + \int_{20}^{40} 2\, dt$$

$$= \left[\frac{t^2}{20}\right]_0^{20} + \left[2t\right]_{20}^{40}$$

$$= \frac{400}{20} + 2 \cdot 40 + (-2) \cdot 20$$

$M = 60$ kg

b) d = average density ; T = total length

$$d = \frac{M}{T} = \frac{60}{40} = 1.5 \text{ kg/m}$$

7) (A) May 2000

$$y = (\tan^{-1} x)^x \to \left(e^{\ln \tan^{-1} x}\right)^x = e^{x \ln(\tan^{-1} x)}$$

derivative =

$$e^{x \ln(\tan^{-1} x)} \cdot \left(\ln(\tan^{-1} x) + (x)\frac{1}{(\tan^{-1} x)} \cdot \frac{1}{1+x^2} \right)$$

$$= (\tan^{-1} x)^x \left(\ln(\tan^{-1}(x)) + \frac{x}{\tan^{-1} x\,(1+x^2)} \right)$$

(B)

$$y = e^{6x} + 6x^e + 6^x$$

$$\frac{\partial y}{\partial x} = 6e^{6x} + 6e x^{e-1} + 6^x \ln 6$$

(C)

$$y = \int_{x^3+x}^{100} \sqrt{p^2 - p}$$

$f(p) = \sqrt{p^2 - p}$

$F'(p) = f(p)$

$$= F(p)\Big|_{x^3+x}^{100} = F(100) - F(x^3+x)$$

$$\Rightarrow \frac{\partial y}{\partial x} = \overset{\text{ct}}{0} - (3x^2+1) \cdot \sqrt{(x^3+x)^2 - x^3 - x} =$$

$$= -(3x^2+1)\sqrt{x^6 + 2x^4 + x^2 - x^3 - x}$$

(D)

$$x^3 y^3 + xy = 10$$

$$3x^2 y^3 \partial x + 3y^2 x^3 \partial y + y\partial x + x\partial y = 0$$

$$\partial x\,(3x^2 y^3 + y) + \partial y\,(3y^2 x^3 + x) = 0$$

$$\partial y\,(3y^2 x^3 + x) = -\partial x\,(3x^2 + y^3 + y)$$

$$\frac{\partial y}{\partial x} = \frac{-(3x^2 + y^3 + y)}{3y^2 x^3 + x}$$

at (2,1)

$$\frac{\partial y}{\partial x} = \frac{-14}{26}$$

May 2000

8.) (A) $\int \frac{4x}{(9x^2+25)^{1/2}}\,dx$ $\quad u = 9x^2+25$, $du = 18x\,dx$ $\Rightarrow dx = \frac{du}{18x}$

$$\int \frac{4x}{u^{1/2}}\frac{du}{18x} = \frac{4}{18}\int u^{1/2}\,du = \frac{4}{18}\left(2(9x^2+25)^{1/2}\right.$$

(B)

$\int \frac{4}{(9x^2+25)^{1/2}}\,dx$ $\quad u = 3x \quad du = 3\,dx \quad dx = \frac{du}{3}$

$$\frac{4}{3}\int \frac{du}{(u^2+5^2)^{1/2}} = \frac{4}{3}\ln\left(u + (5^2+u^2)^{1/2}\right) + C$$

$$= \frac{4}{3}\ln\left(3x + (25+9x^2)^{1/2}\right) + C$$

(C) $\int_5^{21} \frac{x+3}{(3x+1)^{1/2}}\,dx$ $\quad u = 3x+1 \quad x = \frac{u-1}{3}$, $du = 3\,dx \quad dx = \frac{du}{3}$

$$\int_{16}^{64} \frac{\left(\frac{u-1}{3}+3\right)\frac{du}{3}}{u^{1/2}} = \int \frac{\left(\frac{u-1}{9}+1\right)}{u^{1/2}}\,du$$

$$\int \frac{u-1}{9u^{\frac{1}{2}}} + \int \frac{1}{u^{\frac{1}{2}}} = \frac{1}{9}\int u^{\frac{1}{2}} - \frac{1}{9}\int \frac{1}{u^{\frac{1}{2}}} + \int \frac{1}{u^{\frac{1}{2}}} =$$

$$= \frac{1}{9}\int u^{\frac{1}{2}} + \frac{8}{9}\int u^{-\frac{1}{2}} = \frac{1}{9}\left.\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right|_{16}^{64} + \frac{8}{9}\left.\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right|_{16}^{64} =$$

$$= \frac{1088}{27}$$

May 2000

(1) $\displaystyle\int_3^\infty \frac{dx}{(5x+1)^{3/2}}$ $\quad u = 5x+1 \quad dx = \frac{du}{5}$

$du = 5dx$

$$\frac{1}{5}\int \frac{du}{u^{3/2}} \rightarrow \frac{1}{5}(-2)\,u^{-1/2}\Big| = -\frac{2}{5}\left(\frac{1}{(5x+1)^{1/2}}\right)\Big|_3^\infty$$

$$= 0 + \frac{2}{5\sqrt{16}} = \frac{2}{5\sqrt{16}}$$

9.) $$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

$$= x_n - \frac{x^3 - 30x^2 + 200x + 1}{3x^2 - 60x + 200}$$

when $x_n = 10$ and is a root of $x^3 - 30x^2 + 200x$

$$x_{n+1} = 10 - \frac{1}{300 - 600 + 200}$$

$$x_{n+1} = 10 - \frac{1}{100} = \boxed{9.99}$$

10. $\lim\limits_{n\to\infty} P_0 \left(1+\frac{.04}{n}\right)^{nt} = 1^{\infty}$ so use l'Hopital

$$\lim_{n\to\infty} b_0 \left(1+\frac{.04}{n}\right)^{nt} = (1+y)^t P_0 \qquad t=1$$

$$\lim_{n\to\infty} \left(1+\frac{.04}{n}\right)^{n} = (1+y)$$

$$\lim_{n\to\infty} n \ln\left(1+\frac{.04}{n}\right) = \ln(1+y)$$

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{.04}{n}\right)}{\frac{1}{n}} = \ln(1+y)$$

$$\lim_{n\to\infty} \frac{\left(-\frac{.04}{n^2}\right)\frac{1}{1+\frac{.04}{n}}}{-\frac{1}{n^2}} = \ln(1+y)$$

$$\lim_{n\to\infty} \frac{.04}{1+\frac{.04}{n}} = \ln(1+y)$$

$$.04 = \ln(1+y)$$

$$1+y = e^{.04}$$

$$y = e^{.04} - 1 \qquad \text{so APR} = 1 + e^{.04} - 1$$

$$= e^{.04}$$

... # widgets sold × cost of products

$P = w(c - .5)$ $\quad w = w_0 - \frac{\Delta c}{.1}(10000)$ $\quad$ let $\Delta c = x$ $\quad c = 1 + x$

$w = 100000 - x(100000)$

$P = (100{,}000 - 100{,}000x)(1 + x - .5)$

Find $P' = 0$ $\quad \frac{dP}{dx} = (-100{,}000)(.5 + x) + (100{,}000 - 100{,}000x)(1)$ product rule

$= -50{,}000 - 100{,}000x + 100{,}000 - 100{,}000x$

$= 50{,}000 - 200{,}000x = 0$

$x = \frac{50{,}000}{200{,}000} = \frac{1}{4} \Rightarrow \Delta c = \frac{1}{4} = \$.25 \Rightarrow c = 1 + .25 = \boxed{\$1.25}$

Check: profit for $c = \$1$ is \$50,000,

profit for $c = \$1.25 = .75(75000) = \$56{,}250$

~~12) At some c on [1, 5] f(c) will equal~~ $\cancel{\underbrace{\frac{f(5)+f(1)}{2}}_{\text{mean value}} = \frac{\frac{54}{5} + 10}{2} = \frac{104}{10}}$

$\cancel{x^2 + 5x + 4 = \frac{104}{10}x}$

$\cancel{10x^2 - 54x + 40 = 0}$

$\cancel{5x^2 - 27x + 20 = 0}$

$\cancel{x = \frac{27 \pm \sqrt{27^2 - 4(5)(20)}}{10} = \frac{27 \pm \sqrt{329}}{10}}$

12 Mean Value Theorem:

$\exists c \in [1,5]$ so that $f'(c) = \frac{f(5)-f(1)}{5-1}$

$\Leftrightarrow f'(c) = \frac{\frac{25+25+4}{5} - \frac{1+5+4}{1}}{4} = \frac{1}{5}$

$f(x) = \frac{x^2+5x+4}{x} \Rightarrow f'(x) = \frac{(2x+5)\cdot x - (x^2+5x+4)\cdot 1}{x^2} =$

$= \frac{2x^2+5x-x^2-5x-4}{x^2} = \frac{x^2-4}{x^2}$

$\Rightarrow f'(c) = \frac{c^2-4}{c^2} = \frac{1}{5} \Leftrightarrow 5c^2-20 = c^2$

$\Rightarrow c^2 = 5 \Rightarrow c = \pm\sqrt{5}$

$+\sqrt{5} \in [1,5]$ $(\sqrt{5} \approx 2.23)$

$\Rightarrow \exists \boxed{c = \sqrt{5}}$ $\Rightarrow$ satisfies M.V.T.