Name of Student: .....

(Your Instructor's Name: .....)

# Final Examination of Math 1a

**January 23, 2000 (Tuesday)**
**9:15 a.m., Emerson 105 & Harvard 104**

*Instructors: Yum-Tong Siu (course head), Peter Clark, Kim Froyshov*
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| Question | Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 12 | |
| 3 | 20 | |
| 4 | 12 | |
| 5 | 20 | |
| 6 | 16 | |
| 7 | 16 | |
| 8 | 25 | |
| 9 | 12 | |
| 10 | 16 | |
| 11 | 15 | |
| 12 | 16 | |
| Total | 200 | |

- *You have THREE hours to complete this examination.*
- *No calculators are allowed.*
- *No partial credit can be given for unsubstantiated answers.*
- *Use the back of the page if more space is needed for your answer (with an indication that your answer is continued on the back of the page).*

1. Compute the following limits.

(a)

$$\lim_{x \to 2^+} \frac{|x|(x+1)}{(x-2)(x+3)}.$$

(b)

$$\lim_{x \to -\infty} \frac{|x|(x+1)}{(x-2)(x+3)}.$$

(c)

$$\lim_{x \to 0} \frac{1 + x - e^x}{x^2}.$$

(d)

$$\lim_{x \to 0^+} \left[ x^{\sin x} \right].$$

2. Consider the function

$$f(x) = \begin{cases} -|x| & \text{for } x < -\frac{\pi}{2} \\ \cos x & \text{for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{3} \\ x - \frac{\pi}{3} + \frac{1}{2} & \text{for } x > \frac{\pi}{3} \end{cases}.$$

(a) Identify all points $x$ where $f(x)$ is not continuous. Justify your answer.

(b) Identify all points $x$ where $f(x)$ is not differentiable. Justify your answer.

3. Calculate the following derivatives.

(a)

$$\frac{d}{dx}\left[x^2 + \cos x + \ln x\right].$$

(b)

$$\frac{d}{dx}\left[e^x \arctan x\right].$$

(c)

$$\frac{d}{dx}\left[\left(x^2+1\right)^{x^3}\right].$$

(d)

$$\frac{d}{dx}\int_{x^2}^{7}\sin\left(t^3\right)dt.$$

(*Hint:* use the Fundamental Theorem of Calculus.)

4. Let

$$f(x) = e^x + x + \arctan x$$

for $-\infty < x < \infty$.

(a) Prove that $y = f(x)$ is invertible for $-\infty < x < \infty$ by verifying that it is strictly increasing. What is the domain of the inverse function $f^{-1}$ (as a function of $y$)? Justify your answer.

(b) Calcuate the derivative of $f^{-1}$ (as a function of $y$) at $y = 1$. (Note that $y = 1$ when $x = 0$.)

5. For the function

$$f(x) = \frac{x^3 + x^2 - 1}{x^3},$$

answer parts (a) through (h) below.

(*Hint:* $f(x)$ can be written as $1 + \frac{1}{x} - \frac{1}{x^3}$.)

(a) Find any vertical asymptotes for the graph $y = f(x)$.

(b) Find any horizontal asymptotes for the graph $y = f(x)$.

(c) For which values of $x$, if any, is $f(x)$ discontinuous?

(d) Find intervals on which $f$ is increasing and intervals on which $f$ is decreasing.

(e) Find any critical points and determine whether they are relative maxima or minima or neither.

(f) Find intervals where $f$ is concave up and intervals which $f$ is concave down.

(g) Find any points of inflection.

(h) Give a rough sketch of the graph of $y = f(x)$ illustrating all of the features you found in parts (a) through (g).

6. Water is being drained from an upside-down circular cone at the constant rate of 3 cubic feet per minute. When the water level $h$ in the upside-down cone is 9 feet, the water level is decreasing at the rate of 1 foot per minute.

   (a) What is the area $A$ of the circular top of the cone at water level $h$ when $h$ is 9 feet?

   *Hint:* (i) The area $A$ of the circular top of the cone at water level $h$ is proportional to $h^2$ (with the constant of proportionality depending on the cone). (ii) The volume of a cone with base area $A$ and height $h$ is $\frac{1}{3}Ah$.

   (b) How fast is the water level decreasing (in terms of the number of feet per minute) when the water level is 5 feet?

(c) How fast is the area of the circular top of the cone at water level decreasing (in terms of the number of square feet per minute) when the water level is 5 feet?

7. An offshore oil well $W$ is located 4 miles from the closest point $A$ on a straight shoreline. A pipeline is to be contructed from $W$ to a shore point $B$ which is 5 miles from $A$. The pipeline will go along a straight line under water from $W$ to some shore point $P$ between $A$ and $B$ and then go along a straight line over land along the shore from $P$ to $B$. The cost of of laying the pipeline is 5 million dollars a mile under water but is only 3 million dollars over land.

   (a) How far away should $P$ be from $A$ (in miles) to minimize the construction cost of the pipeline?

   (b) What is the minimum construction cost?

8. Compute the following indefinite and definite integrals.

(a)

$$\int \left(x^2 + e^x\right) dx.$$

(b)

$$\int \sin^3 x \, \cos x \, dx.$$

(c)

$$\int \frac{\arctan x}{1+x^2}\,dx.$$

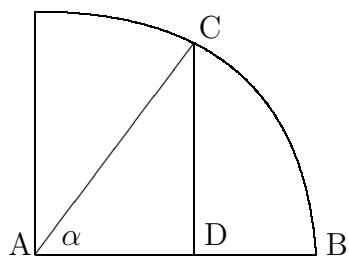
(d)

$$\int_{-1}^{2} \left|1-e^{-x}\right| dx.$$

(e)

$$\int_{\frac{3}{2}}^{2} \sqrt{2x - x^2}dx$$

[*Hint:* Complete the square for $2x - x^2$, write the integral as the area of the sector $ABC$ minus the area of the triangle $ADC$. The area of a sector (or a fan) of angle $\alpha$ (measured in radians) in a disk of radius $r$ is $\frac{1}{2}r^2\alpha$.]



9. (a) Find

$$I = \int_1^{10} \frac{dx}{x}$$

(b) Write down the Riemann sum $R$ of

$$\int_1^{10} \frac{dx}{x}$$

with 9 subintervals, using right end-points.

(c) Write down the Riemann sum $L$ of

$$\int_1^{10} \frac{dx}{x}$$

with 9 subintervals, using left end-points.

(d) Which of the three numbers $I$, $R$, $L$ is the largest and which is the smallest? Justify your answer.

10. A creature moves along the $x$-axis in such a way that its acceleration is given by $a(t) = 2 - 6t$. The time variable $t$ is allowed to take only nonnegative values. When the creature starts out at $t = 0$, it is at the position $x = 2$ and has velocity 1 at that moment.

    (a) Where will the creature be when $t = 2$?

    (b) At what time $t$ is the creature closest to the point $(4, 1)$ in the $xy$-plane? What is the distance between the creature and the point $(4, 1)$ at that time?

11. A ball is thrown directly upward with an initial velocity of 128 feet per second and is released from a point that is 144 feet above ground. Assume that the free-fall model applies and that the acceleration due to gravity is 32 feet per second per second (*i.e.,* the ball falls with constant acceleration due to gravity of 32 feet per second per second).

    (a) In the terms of the number of feet above ground, what is the highest point which the ball reaches?

    (You may leave your answer as the product of two integers.)

    (b) What is the velocity of the ball when it reaches the highest point?

(c) How much time elapses after the ball is thrown and before it reaches the ground on its fall?

12. (a) State the following three theorems, including all necessary conditions.

    (i) The Intermediate-Value Theorem.

    (ii) The Extreme-Value Theorem.

    (iii) The Mean-Value Theorem.

    (b) Two particles move along a straight line during the time interval from $t = a$ to $t = b$. Suppose the two particles have the same average velocity for that time interval. Use one of the three theorems in (a) to show that there is a certain point in that time interval when both particles have the same instantaneous velocity.

(c) A hare is racing a tortoise along a straight line. At the start of the race the hare is 10 meters behind the tortoise and at the end of the race the hare is 20 meters ahead of the tortoise. Use one of the three theorems in (a) to show that there is a certain point in the race when the hare is exactly 5 meters ahead of the tortoise.

(d) Let $f(x) = e^x - e^{x^2} \sin x$. Use one of the three theorems in (a) to show that there is some point $a$ in the interval $[-50, 100]$ such that $f(a) \geq f(x)$ for $-50 \leq x \leq 100$.