Name: ______________________________ ID#: ____________________

Solutions to
Midterm Examination II

Math 1a
Introduction to Calculus

26 April 2004

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

> *Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.*
>
> —*Handbook for Students*

1. (20 Points) *Find the derivatives of the following functions.*

(i) $f(x) = x^7 - 6x^4 + 3x^3 + x - 17$

Solution. We use the Power Rule and the Sum Rule:

$$f'(x) = 7x^6 - 4 \cdot 6x^3 + 3 \cdot 3x^2 + 1 = 7x^6 - 24x^3 + 9x^2 + 1.$$

□

(ii) $f(x) = x^{x \ln x}$

Solution. We use logarithmic differentiation: if $y = x^{x \ln x}$, then

$$\begin{aligned} \ln y &= (x \ln x)(\ln x) = x(\ln x)^2; \\ \frac{1}{y}\frac{dy}{dx} &= x(2 \ln x)\left(\frac{1}{x}\right) + (\ln x)^2; \\ \frac{dy}{dx} &= x^{x \ln x}(\ln x)\left(2 + \ln x\right). \end{aligned}$$

□

(iii) $f(x) = \dfrac{x^3}{x^3 - 1}$

Solution. We use the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(x^3 - 1)(3x^2) - x^3(3x^2)}{(x^3 - 1)^2} \\ &= \frac{-3x^2}{(x^3 - 1)^2}. \end{aligned}$$

□

(iv) $f(x) = \sin(\sin(\sin(x)))$

Solution. This is just several applications of the chain rule:

$$f'(x) = \cos(\sin(\sin x))\cos(\sin x)\cos(x).$$

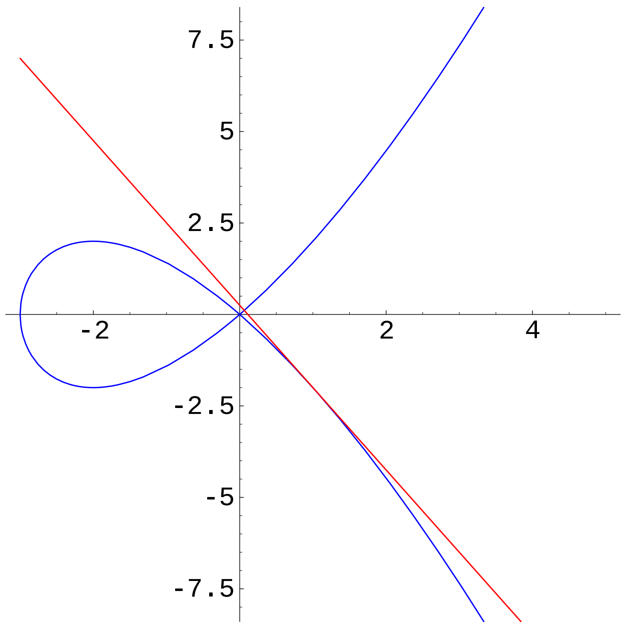
□

______ / 20

2. (15 Points) *Consider the curve*

$$y^2 = x^3 + 3x^2.$$

Find the equation of the tangent line to this curve at the point $(1, -2)$ *in slope-intercept* $(y = mx + b)$ *form.*



Solution. We use implicit differentiation:

$$2y\frac{dy}{dx} = 3x^2 + 6x;$$
$$\frac{dy}{dx} = \frac{3x^2 + 6x}{2y}.$$

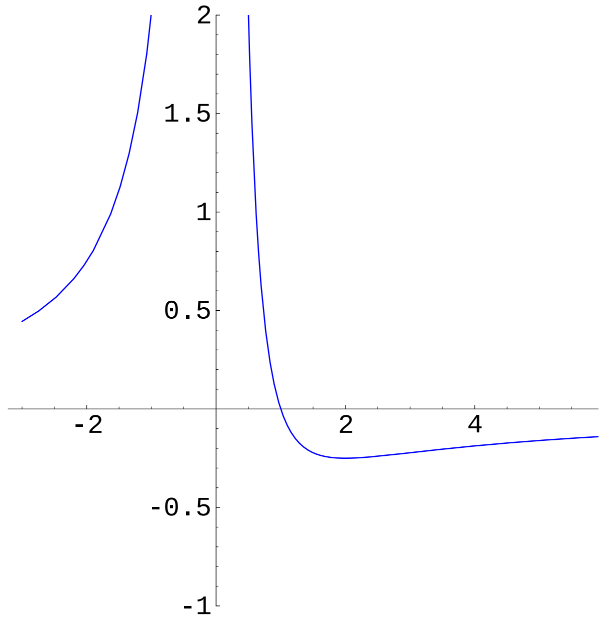
So at the point $(1, -2)$ the slope of the line tangent to the curve is $-\frac{9}{4}$. Therefore the *point-slope* form of the line is

$$y + 2 = -\frac{9}{4}(x - 1),$$

which, in slope-intercept form, is $y = -\frac{9}{4}x + \frac{1}{4}$. □

______ / 15

3. (25 Points) *Graph the function* $f(x) = \dfrac{1-x}{x^2}$ *and answer the questions on the following page.*



(i) *What is/are the critical point(s)? Give their x- and y-values. Label them as local or global maxima or minima.*

Solution. Using the fact that

$$f(x) = \frac{1}{x^2} - \frac{1}{x};$$
$$f'(x) = \frac{-2}{x^3} + \frac{1}{x^2} = \frac{x-2}{x^3}.$$

We have a critical point at $(2, 1/4)$. Now

$$f''(x) = \frac{6}{x^4} - \frac{2}{x^3} = \frac{6-2x}{x^4},$$

we have $f''(1) = 6 > 0$, and so this point is a minimum. It's a global minimum because $\lim_{x\to 0} f(x) = \infty$, and there are no other critical points. □

(ii) *On what intervals is the function increasing? decreasing?*

Solution. We can make a table:

interval	$\frac{1}{x^3}$	$x-2$	$f'(x)$
$-\infty < x < 0$	$-$	$-$	$+$
$0 < x < 2$	$+$	$-$	$-$
$2 < x < \infty$	$+$	$+$	$+$

So f is *increasing* on $(-\infty, 0)$ and $(2, \infty)$, and decreasing on $(0, 2)$. □

(iii) On what intervals is the function concave up? concave down?

Solution. We can make a table:

interval	$\frac{1}{x^4}$	$6-2x$	$f'(x)$
$-\infty < x < 0$	$+$	$+$	$+$
$0 < x < 3$	$+$	$+$	$+$
$3 < x < \infty$	$+$	$-$	$-$

So f *concave up* on $(-\infty, 0)$ and $(0, 3)$, and concave down on $(3, \infty)$. □

(iv) Where are the inflection points, if any?

Solution. f switches from concave up to concave down at $(3, 2/9)$, and therefore has an inflection point there. □

(v) What asymptotes does the graph have, if any?

Solution. There is a vertical asymptote where the denominator vanishes, namely at $x = 0$. Since the limit of the numerator at zero is 1, whilst the denominator goes to zero but remains positive (remember it's x^2), the limit of the quotient is ∞.

There is a horizontal asymptote at the line $y = 0$ because

$$\lim_{x \to \pm\infty} f(x) = 0.$$

□

______ / 25

4. (15 Points) *Find the following limits.*

(i) $\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{x - \frac{\pi}{2}}$

Solution. I had intended for L'Hôpital's Rule to be used here; however, since the numerator approaches ∞ and the denominator 0 at the point in question, the form is not indeterminate (the limit is $-\infty$). So I disregarded this question. □

(ii) $\lim_{x \to 0} \frac{\cos x}{x}$

Solution. Here L'Hôpital's Rule does *not* apply, because the numerator goes to 1, not 0, as $x \to 0$. So the limit is ∞. If you take the derivative of the numerator and denominator, you will get $\frac{\sin x}{1} \to 0$, which is incorrect. □

(iii) $\lim_{x \to 0} \frac{\sqrt{1+x} - \frac{1}{2}x - 1}{x^2}$

Solution. Here we must use L'Hôpital's Rule twice:

$$\begin{aligned} \lim_{x \to 0} \frac{\sqrt{1+x} - \frac{1}{2}x - 1}{x^2} &\overset{\text{H}}{=} \lim_{x \to 0} \frac{\frac{1}{2}(1+x)^{-1/2} - \frac{1}{2}}{2x} \\ &\overset{\text{H}}{=} \lim_{x \to 0} \frac{-\frac{1}{4}(1+x)^{-3/2}}{2} = -\frac{1}{8}. \end{aligned}$$

□

______ / 15

5. (15 Points) *Let* $f(x) = \frac{a+x}{2} - \sqrt{ax}$*, on the interval* $(0, \infty)$.

(a) (6 points) Find $f'(x)$.

Solution. Remember that a is a constant, so the derivative is

$$f'(x) = \tfrac{1}{2} - \frac{a}{2\sqrt{ax}} = \frac{1}{2}\left(1 - \frac{\sqrt{a}}{\sqrt{x}}\right).$$

Many had trouble using the chain rule to get the derivative of the square root term. □

(b) (6 points) Show that the point $(a, 0)$ *is the global minimum of* f.

Solution. We see from the above expression for f' that $f'(a) = 0$, and this is the only critical point for f. When $x < a$, $f'(x)$ is negative, so f is decreasing. When $x > a$, $f'(x)$ is positive, so f is increasing. This means f achieves its global minimum at a.

Note that we need not worry about the $\sqrt{x}$ in the denominator of f' since f is only defined for *positive* x. □

(c) (3 points) Explain why the inequality

$$\sqrt{ax} \leq \frac{a+x}{2}$$

follows from (b). We have proven the famous arithmetic mean–geometric mean *inequality.*

Solution. In (b) we showed that $(a, 0)$ was the global minimum of f. This means that for all $x > 0$, $f(x) \geq f(a)$. In other words,

$$\frac{a+x}{2} - \sqrt{ax} \geq 0.$$

□

______ / 15

6. (10 Points) *Chris and Mitch are undergraduates at Pacific Tech and are trying to pop an enormous tin of popcorn, which happens to be located inside the house of the professor in charge of their laboratory. They fire a laser from an airplane in very high altitude so that it hits the tin of popcorn and causes it to gradually heat.*

As the popcorn pops, it fills out its foil container which attains a spherical shape. After one minute of popping, the sphere of popcorn is observed to be six feet in diameter and growing at the rate of one inch of diameter per second.

What is the rate at which the popcorn is being popped (i.e., what is the rate of change of the volume of popcorn) at this time? Give your answer in $\frac{in^3}{sec}$.

Remember the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Solution. Differentiating the volume formula of a sphere with respect to time gives

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

We are given that when $r = 3$ feet, $\frac{dr}{dt} = \frac{1}{2}\frac{\text{in.}}{\text{sec.}}$. We must be consistent about units, and since we are asked to give a final answer in terms of $\frac{\text{in}^3}{\text{sec}}$, we convert the radius to 36 inches. Our answer is

$$\frac{dV}{dt} = 4\pi(36)^2\left(\tfrac{1}{2}\right) = 2592\pi\frac{\text{in}^3}{\text{sec}}.$$

□

□ Check the box if you know what movie this problem comes from.

Solution. Only a handful knew this problem was based on *Real Genius*, one of Val Kilmer's best movies in the pre-*Top Gun* era. □

______ / 10

(This page intentionally left blank.)