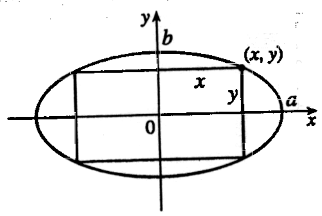
Math 1A Fall 2001: Section 4.6, Assignment 2 Solutions

18.



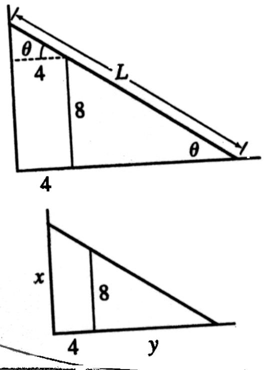
The area of the rectangle is $(2x)(2y) = 4xy$. Now $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ gives $y = \frac{b}{a}\sqrt{a^2 - x^2}$, so we maximize $A(x) = 4\frac{b}{a}x\sqrt{a^2 - x^2}$.

$$A'(x) = \frac{4b}{a}\left[x \cdot \tfrac{1}{2}(a^2 - x^2)^{-1/2}(-2x) + (a^2 - x^2)^{1/2} \cdot 1\right]$$

$$= \frac{4b}{a}(a^2 - x^2)^{-1/2}\left[-x^2 + a^2 - x^2\right] = \frac{4b}{a\sqrt{a^2 - x^2}}\left[a^2 - 2x^2\right]$$

So the critical number is $x = \frac{1}{\sqrt{2}}a$, and this clearly gives a maximum. Then $y = \frac{1}{\sqrt{2}}b$, so the maximum area is $4\left(\frac{1}{\sqrt{2}}a\right)\left(\frac{1}{\sqrt{2}}b\right) = 2ab$.

22.



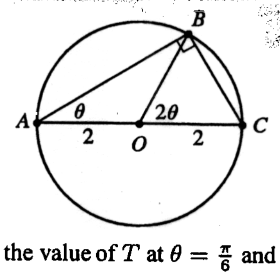
$L = 8\csc\theta + 4\sec\theta,\ 0 < \theta < \frac{\pi}{2}$,

$\frac{dL}{d\theta} = -8\csc\theta\cot\theta + 4\sec\theta\tan\theta = 0$ when $\sec\theta\tan\theta = 2\csc\theta\cot\theta \Leftrightarrow \tan^3\theta = 2 \Leftrightarrow \tan\theta = \sqrt[3]{2} \Leftrightarrow \theta = \tan^{-1}\sqrt[3]{2}$.

$dL/d\theta < 0$ when $0 < \theta < \tan^{-1}\sqrt[3]{2}$, $dL/d\theta > 0$ when $\tan^{-1}\sqrt[3]{2} < \theta < \frac{\pi}{2}$, so L has an absolute minimum when $\theta = \tan^{-1}\sqrt[3]{2}$, and the shortest ladder has length

$$L = 8\frac{\sqrt{1 + 2^{2/3}}}{2^{1/3}} + 4\sqrt{1 + 2^{2/3}} \approx 16.65 \text{ ft.}$$

28.



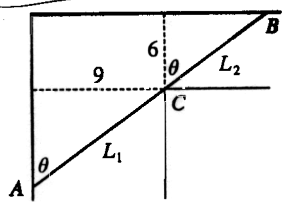
In isosceles triangle AOB, $\angle O = 180° - \theta - \theta$, so $\angle BOC = 2\theta$. The distance rowed is $4\cos\theta$ while the distance walked is the length of arc $BC = 2(2\theta) = 4\theta$. The time taken is given by

$$T(\theta) = \frac{4\cos\theta}{2} + \frac{4\theta}{4} = 2\cos\theta + \theta, \quad 0 \le \theta \le \tfrac{\pi}{2}.$$

$$T'(\theta) = -2\sin\theta + 1 = 0 \Leftrightarrow \sin\theta = \tfrac{1}{2} \Rightarrow \theta = \tfrac{\pi}{6}.$$

Check the value of T at $\theta = \frac{\pi}{6}$ and at the endpoints of the domain of T; that is, $\theta = 0$ and $\theta = \frac{\pi}{2}$. $T(0) = 2$, $T\left(\frac{\pi}{6}\right) = \sqrt{3} + \frac{\pi}{6} \approx 2.26$, and $T\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \approx 1.57$. Therefore, the minimum value of T is $\frac{\pi}{2}$ when $\theta = \frac{\pi}{2}$; that is, the woman should walk all the way. Note that $T''(\theta) = -2\cos\theta < 0$ for $0 \le \theta < \frac{\pi}{2}$, so $\theta = \frac{\pi}{6}$ gives a maximum time.

36.



Paradoxically, we solve this maximum problem by solving a minimum problem. Let L be the length of the line ACB going from wall to wall touching the inner corner C. As $\theta \to 0$ or $\theta \to \frac{\pi}{2}$, we have $L \to \infty$ and there will be an angle that makes L a minimum. A pipe of this length will just fit around the corner.

From the diagram, $L = L_1 + L_2 = 9\csc\theta + 6\sec\theta \Rightarrow dL/d\theta = -9\csc\theta\cot\theta + 6\sec\theta\tan\theta = 0$ when $6\sec\theta\tan\theta = 9\csc\theta\cot\theta \Leftrightarrow \tan^3\theta = \frac{9}{6} = 1.5 \Leftrightarrow \tan\theta = \sqrt[3]{1.5}$. Then $\sec^2\theta = 1 + \left(\frac{3}{2}\right)^{2/3}$ and $\csc^2\theta = 1 + \left(\frac{3}{2}\right)^{-2/3}$, so the longest pipe has length $L = 9\left[1 + \left(\frac{3}{2}\right)^{-2/3}\right]^{1/2} + 6\left[1 + \left(\frac{3}{2}\right)^{2/3}\right]^{1/2} \approx 21.07$ ft.

Or, use $\theta = \tan^{-1}\left(\sqrt[3]{1.5}\right) \approx 0.852 \Rightarrow L = 9\csc\theta + 6\sec\theta \approx 21.07$ ft.

40. Let x be the distance from the observer to the wall. Then, from the given figure,

$$\theta = \tan^{-1}\left(\frac{h+d}{x}\right) - \tan^{-1}\left(\frac{d}{x}\right),\ x > 0 \Rightarrow$$

$$\frac{d\theta}{dx} = \frac{1}{1 + [(h+d)/x]^2}\left[-\frac{h+d}{x^2}\right] - \frac{1}{1 + (d/x)^2}\left[-\frac{d}{x^2}\right] = -\frac{h+d}{x^2 + (h+d)^2} + \frac{d}{x^2 + d^2}$$

$$= \frac{d[x^2 + (h+d)^2] - (h+d)(x^2 + d^2)}{[x^2 + (h+d)^2](x^2 + d^2)} = \frac{h^2 d + hd^2 - hx^2}{[x^2 + (h+d)^2](x^2 + d^2)} = 0 \Leftrightarrow$$

$$hx^2 = h^2 d + hd^2 \Leftrightarrow x^2 = hd + d^2$$