Section 7.6

10. $x(1+x^3) = x + x^4 = f(x)$. The antiderivative of $f(x)$
is just $F(x) = \frac{1}{2}x^2 + \frac{1}{5}x^5$. By FTC,

$$\int_{-1}^{2} f(x)\,dx = F(2) - F(-1).$$ That is,

$$\int_{-1}^{2} x(1+x^3)\,dx = \frac{1}{2}(2)^2 + \frac{1}{5}(2)^2 - \left(\frac{1}{2}(-1)^2 + \frac{1}{5}(-1)^5\right)$$

$$= 2 + \frac{4}{5} - \frac{1}{2} + \frac{1}{5} = \boxed{\frac{5}{2}}$$

12. $\frac{1}{x^6} = x^{-6} = f(x)$. The antiderivative of $f(x)$
is just $F(x) = -\frac{1}{5}x^{-5}$. By FTC.

$$\int_{1}^{2} f(x)\,dx = F(2) - F(1).$$

Thus, $$\int_{1}^{2} \frac{1}{x^6}\,dx = -\frac{1}{5}(2)^{-5} - \left(-\frac{1}{5}(1)^{-5}\right) = -\frac{1}{5\cdot 32} + \frac{1}{5}$$

$$= \frac{32}{5\cdot 32} - \frac{1}{5\cdot 32} = \boxed{\frac{31}{160}}$$

14. $5x^{2/3} - 4x^{-2} = f(x)$. The antiderivative of $f(x)$ is just
$F(x) = 5\left(\frac{3}{5}x^{5/3}\right) - 4\left(-x^{-1}\right)$

$$= 3x^{5/3} + 4x^{-1}$$ By FTC,

$$\int_{1}^{8} f(x)\,dx = F(8) - F(1).$$ Thus, $$\int_{1}^{8} 5x^{2/3} - 4x^{-2}\,dx = 3(8)^{5/3} + 4(8)^{-1} - 3(1)^{5/3} + 4(1)^{-1}$$

$$= 3\cdot 2^5 + \frac{1}{2} - 3 + 4 = 96 + \frac{1}{2} + 1 = \boxed{\frac{195}{2}}$$

20. $\frac{1}{2x} = \frac{1}{2}x^{-1} = f(x)$. The antiderivative of $f(x)$ is just

$F(x) = \frac{1}{2}\ln x$. So, by FTC,

$\int_{\frac{1}{2}}^{1} f(x)\,dx = F(1) - F(\frac{1}{2})$. Thus, $\int_{\frac{1}{2}}^{1} \frac{1}{2x}\,dx = \frac{1}{2}\ln 1 - \frac{1}{2}\ln\frac{1}{2}$

$= -\frac{1}{2}\ln\frac{1}{2} = \boxed{\ln\sqrt{2}}$

27a) Let $f(x) = |2x-3|$. We can't write down an antiderivative for this function. However, $f(x) = \begin{cases} 2x-3, & x \geq \frac{3}{2} \\ 3-2x, & x \leq \frac{3}{2} \end{cases}$

So, $\int_0^2 |2x-3|\,dx = \int_0^{\frac{3}{2}} |2x-3|\,dx + \int_{\frac{3}{2}}^{2} |2x-3|\,dx$

by Theorem 7.5.5.

$= \int_0^{\frac{3}{2}} 3-2x\,dx + \int_{\frac{3}{2}}^{2} 2x-3\,dx$

The antiderivative of $3-2x$ is $3x - x^2$; the antiderivative of $2x-3$ is $x^2 - 3x$. So, by FTC, this is just equal to

$= 3(\frac{3}{2}) - (\frac{3}{2})^2 - (3(0) - 0^2) + 2^2 - 3\cdot 2 - ((\frac{3}{2})^2 - 3(\frac{3}{2}))$

$= \frac{9}{2} - \frac{9}{4} + 4 - 6 - \frac{9}{4} + \frac{9}{2}$

$= 9 - \frac{9}{2} - 2 = \frac{9}{2} - 2 = \boxed{\frac{5}{2}}$

b) $\int_0^{3\pi/4} |\cos x|\, dx$. Note that $\cos x \geq 0$ for $x \in [0, \frac{\pi}{2}]$, but $\cos x \leq$ for $x \in [\frac{\pi}{2}, \frac{3\pi}{4}]$. Thus,

$$|\cos x| = \begin{cases} \cos x, & x \in [0, \frac{\pi}{2}] \\ -\cos x, & x \in [\frac{\pi}{2}, \frac{3\pi}{4}] \end{cases}$$

So, $\int_0^{\frac{3\pi}{4}} |\cos x|\, dx = \int_0^{\frac{\pi}{2}} \cos x\, dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} -\cos x\, dx$.

The antiderivative of $\cos x$ is $\sin x$. Thus,

$$\int_0^{\frac{3\pi}{4}} |\cos x|\, dx = \sin\frac{\pi}{2} - \sin 0 - \left(\sin\frac{3\pi}{4} - \sin\frac{\pi}{2}\right)$$
$$= 1 - 0 - \left(-\frac{\sqrt{2}}{2}\right) + 1 = \boxed{2 - \frac{\sqrt{2}}{2}}$$

46. a) By FTC Part 2, $\cancel{\frac{d}{dx}\left[\int^x \sin(\sqrt{t})\, dt\right] =}$

$$\frac{d}{dx}\left[\int_0^x \frac{1}{1+\sqrt{t}}\, dt\right] = \boxed{\frac{1}{1+\sqrt{x}}}$$

b) Similarly, $\frac{d}{dx}\left[\int_1^x \ln t\, dt\right] = \boxed{\ln x}$

49. a) $F(2) = \int_2^2 \sqrt{3t^2+1}\, dt = \boxed{0}$ because integrating over an interval of length 0!

b) $F'(x) = \frac{d}{dx}\left[\int_2^x \sqrt{3t^2+1}\, dt\right] = \sqrt{3x^2+1}$ by FTC.

$$F'(2) = \sqrt{3 \cdot 4 + 1} = \boxed{\sqrt{13}}$$

c) $F''(x) = 6x\left(\frac{1}{2}(3x^2+1)^{-\frac{1}{2}}\right)$ by the chain rule.

So $F''(2) = 3 \cdot 2 \cdot \frac{1}{\sqrt{13}} = \boxed{\frac{6}{\sqrt{13}}}$

51. a) To see where the extrema of this function are, we must see where its derivative is 0 or undefined.

$$F'(x) = \frac{d}{dx}\left[\int_0^x \frac{t-3}{t^2+7}\,dt\right] = \frac{x-3}{x^2+7} \quad \text{by FTC.}$$

$F'(3) = 0$, otherwise $F'(x)$ is nonzero and well-defined.

To see if it's a minimum or a maximum, we use the second derivative test: $F''(x) = \dfrac{(x^2+7)\cdot 1 - (x-3)(2x)}{(x^2+7)^2}$

$$= \frac{x^2+7-2x^2+6x}{(x^2+7)^2} = \frac{7-x^2+6x}{(x^2+7)^2}$$

$F''(3) = \dfrac{7-9+18}{(9+7)} > 0 \Rightarrow F(x)$ is concave up at $x=3$

$\Rightarrow x=3$ is a local minimum. Since it is the only critical point, $x=3$ is the global minimum of $F(x)$ over $\mathbb{R}$.

b) $F'(x) > 0$ when $x > 3$, since the denominator is always positive.

Thus, $F(x)$ is strictly increasing on $(3, \infty)$.

$F'(x) < 0$ when $x < 3$. So, $F(x)$ is strictly decreasing on $(-\infty, 3)$

~~c) $F''(x) > 0$ when $x^2 < 10 \Rightarrow -\sqrt{10} < x < \sqrt{10}$, since the denominator is always positive~~

~~So, $F(x)$ is concave up on $(-\sqrt{10}, \sqrt{10})$~~

~~$F''(x) < 0$ when $x^2 > 10 \Rightarrow x < -\sqrt{10}$ or $x > \sqrt{10}$~~

~~Thus, $F(x)$ is concave down on $(-\infty, -\sqrt{10}) \cup (\sqrt{10}, \infty)$.~~

c) $7 + 6x - x^2 = (7-x)(1+x)$. This is greater than zero if $7-x$ and $1+x$ have the same sign: i.e.,

if $x < 7$ and $x > -1$. Since $F''(x)$ has the same sign as $7+6x-x^2$ (the denominator of $F''(x)$ is always positive), this means that $F(x)$ is concave up on $(-1, 7)$. Thus, $F(x)$ is concave down on $(-\infty, -1) \cup (7, \infty)$.

56. a) $\int_{-\pi}^{\pi} \sin x \, dx = -\cos\pi + \cos -\pi = 0$. We're looking for x^* such than $\sin x^* = 0(\pi - (-\pi)) = 0$, and $x^* \in (-\pi, \pi)$.

In this case, $x^* = 0$, since $\sin 0 = 0$.

x^* is the point at which the function achieves its average value of the interval. – The average value of $\sin x$ over $[-\pi, \pi]$ is 0, which is achieved at $x^* = 0$.

b) $\int_1^3 \frac{1}{x^2} \, dx = \int_1^3 x^{-2} \, dx = -(3)^{-1} + (1)^{-1} = -\frac{1}{3} + 1 = \frac{2}{3}$.

We're looking for an $x^* \in (1,3)$ such that

$$\frac{1}{(x^*)^2} = \frac{2}{3}(3-1) = \frac{4}{3} \Rightarrow (x^*)^2 = \frac{3}{4} \Rightarrow \boxed{x^* = \frac{\sqrt{3}}{2}},$$

which does indeed lie in our interval.

x^* has exactly the same interpretation as in part (a).