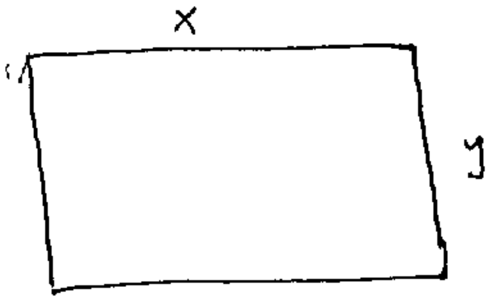
Homework #18

Section 6.2

5.



Area $= xy$.

Total cost $= (3x)\cdot 2 + (2y)\cdot 2 = 6000$.

$\Rightarrow \quad 6x = 6000 - 4y$

$\Rightarrow \quad x = 1000 - \frac{2}{3}y$

$$\Rightarrow \quad \text{Area} = \left(1000 - \frac{2}{3}y\right)y = 1000y - \frac{2}{3}y^2$$

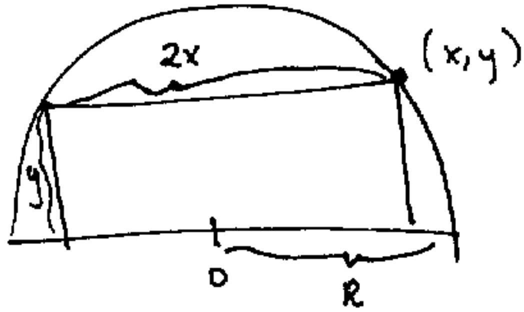
$$\frac{d}{dy}(\text{Area}) = 1000 - \frac{4}{3}y = 0 \quad \text{when}$$

$$\frac{4}{3}y = 1000 \Rightarrow y = \frac{3}{4}(1000) = 750.$$

$$\rightarrow \quad x = 1000 - \frac{2}{3}(750) = 500$$

The rectangle with the largest ~~dimensions~~ area has dimensions 750 ft (of the cheaper fencing) by 500 ft (of the heavy-duty fencing).

10.



For convenience, we impose a coordinate system on the diagram. Here, the area of the rectangle is $2xy$.

The point (x,y) lies on the semicircle, so we know that $x^2+y^2 = R^2$

$$\Rightarrow \quad \frac{d}{dy}\left(x^2+y^2\right) = \frac{d}{dy}\left(R^2\right) = 0$$

$$\Rightarrow \quad 2x \cdot \frac{dx}{dy} + 2y = 0$$

$$\Rightarrow \quad \frac{dx}{dy} = -\frac{y}{x}.$$

$$\frac{d}{dy}(\text{Area}) = 2x + 2y \cdot \frac{dx}{dy} \quad \text{by the product rule}$$

$$= 2x + 2y\left(-\frac{y}{x}\right)$$

$$= 2x - \frac{2y^2}{x}$$

$$= 2x - 2\,\frac{R^2-x^2}{x} = 0$$

when $2x = 2\,\dfrac{R^2-x^2}{x}$

$$\Rightarrow \quad x^2 = R^2 - x^2$$

$$\Rightarrow \quad 2x^2 = R^2$$

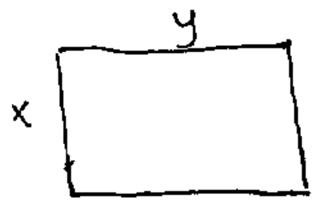
$$\Rightarrow \quad x = \sqrt{\tfrac{1}{2}R^2} = \frac{1}{\sqrt{2}}R$$

$$x^2+y^2=R^2 \Rightarrow \frac{1}{2}R^2+y^2=R^2$$
$$\Rightarrow y = \frac{1}{2}R^2$$
$$\Rightarrow y=\sqrt{\tfrac{1}{2}R^2} = \frac{1}{\sqrt{2}}R^2.$$

So, the rectangle with largest area has

$$(x,y) = \left(\frac{1}{\sqrt{2}}R, \frac{1}{\sqrt{2}}R\right).$$

11.



$$\text{Area} = xy = 3200$$
$$\text{Cost} = x\cdot 2 + y\cdot 1 = 2x+y$$

$$xy = 3200 \Rightarrow x = \frac{3200}{y}$$

$$\Rightarrow \text{Cost} = \frac{6400}{y} + y.$$

$$\frac{d}{dy}(\text{cost}) = -\frac{6400}{y^2} + 1 = 0$$

$$\text{when } \frac{6400}{y^2} = 1$$

$$\Rightarrow y^2 = 6400$$

$$\Rightarrow y = 80$$

$$\Rightarrow x = \frac{3200}{80} = 40$$

$\Rightarrow$ the dimensions that cost the least are 80 ft (cheaper fencing) by 40 ft (more expensive fencing)

16. 

$$\text{Perimeter} = 2x + 2r + \tfrac{1}{2}(2\pi r)$$
$$= 2x + r(2+\pi) = p, \text{ where } p \text{ is some constant.}$$

We want to maximize $\text{area} = 2xr + \frac{1}{2}(\pi r^2)$.

We know that $x = \dfrac{p - r(2+\pi)}{2}$

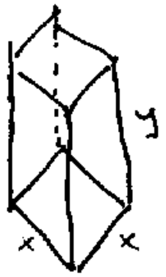
$$\Rightarrow \text{Area} = r\big(p - r(2+\pi)\big) + \tfrac{1}{2}\pi r^2$$
$$= p\cdot r + r^2\left(\tfrac{1}{2}\pi - 2 - \pi\right)$$
$$= p\cdot r - r^2\left(2 + \tfrac{1}{2}\pi\right)$$

$$\frac{d}{dr}(\text{Area}) = p - 2r\left(2 + \tfrac{1}{2}\pi\right)$$
$$= p - r(4+\pi) = 0$$

when $r(4+\pi) = p$

$$\Rightarrow \boxed{r = \frac{p}{4+\pi}}$$

is the radius that gives you the best area for a given p.

20. 

$$\text{Area} = x^2 y = 2250.$$

$$\text{Total cost} = x^2 \cdot 2 \cdot 2 + xy \cdot 4 \cdot 3.$$

$$= 4x^2 + 12xy$$

$$y = \frac{2250}{x^2}$$

$$\Rightarrow \quad \text{Cost} = 4x^2 + \frac{12x \cdot 2250}{x^2}$$

$$= 4x^2 + \frac{12 \cdot 2250}{x}$$

$$\frac{d}{dx}(\text{cost}) = 8x - \frac{12 \cdot 2250}{x^2} = 0$$

$$\text{when} \quad 8x = \frac{12 \cdot 2250}{x^2}$$

$$\Rightarrow \quad 8x^3 = 12 \cdot 2250$$

$$\Rightarrow \quad x^3 = \frac{3}{2} \cdot 2250$$

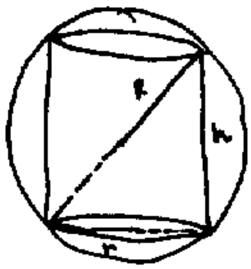
$$= \frac{6750}{2} = 3375 = 27 \cdot 125$$

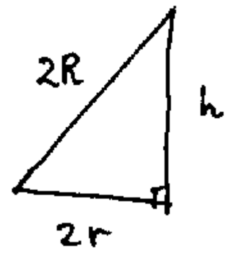
$$\Rightarrow \quad x = 3 \cdot 5 = 15.$$

$$\Rightarrow \quad y = \frac{2250}{15^2} = \frac{2250}{225} = 10.$$

$\Rightarrow$ dimensions of the best box are $\boxed{15 \times 15 \times 10.}$

24





this is a right triangle — the opposite points on the cylinder are on opposite points of the sphere, and the cylinder makes a right angle. So we can say $(2r)^2 + h^2 = (2R)^2$,

or $4r^2 + h^2 = 4R^2$, where R is some fixed constant.

The area of the cylinder is $\pi r^2 h$.

But, we know that $4r^2 = 4R^2 - h^2$

$$\Rightarrow \quad r^2 = \frac{4R^2 - h^2}{4}$$

$$\Rightarrow \quad \text{Area} = \pi \left(\frac{4R^2 - h^2}{4} \right) h$$

$$= \frac{\pi}{4} \left(4R^2 h - h^3 \right)$$

$$\frac{d}{dh}(\text{area}) = \frac{\pi}{4} \left(4R^2 - 3h^2 \right) = 0$$

when $4R^2 = 3h^2 \Rightarrow \boxed{h = \sqrt{\frac{4}{3}}\, R}$

$$\Rightarrow \quad 4r^2 + \frac{4}{3} R^2 = 4R^2$$

$$\Rightarrow \quad 4r^2 = \frac{8}{3} R^2 \Rightarrow r^2 = \frac{2}{3} R^2$$

$$\Rightarrow \boxed{r = \sqrt{\frac{2}{3}}\, R}$$

44. a) revenue $= R(x) =$ total sales $=$

(# of units sold) $\cdot$ (price per unit)

Thus, $R(x) = x \cdot p$

Since $p = 1000 - x$, $\boxed{R(x) = x(1000-x)}$

b) Profit = Revenue − Cost

So, $P(x) = x(1000-x) - (3000 + 20x)$

$$= 980x - x^2 - 3000$$

c) First, let's check the endpoints:

$$P(0) = 0 - 0 - 3000 = -3000.$$

$$P(500) = 490000 - 250000 - 3000$$

$$= 237000.$$

Now, let's look for local extrema:

$P'(x) = 980 - 2x = 0$ when

$2x = 980 \Rightarrow x = 490$

Note that this is actually within the production capacity of the company.

$P(490) = 480200 - 240100 - 3000$

$= 237100$

So, the maximum profit occurs that $\boxed{x = 490}$

d) $P(490) = 237100$, so the maximum profit is $\boxed{\$237100}$

e) $p = 1000 - x = 1000 - 490 = \boxed{\$510}$.