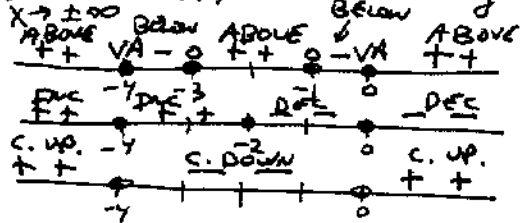


MATH 1a EXAM 2 REVIEW PROBLEM SOLUTIONS

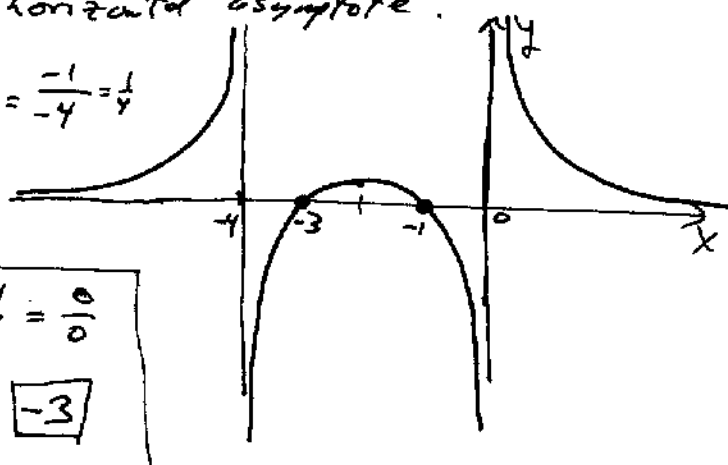
①  $f(x)=0$  when  $x^2+4x+3=(x+3)(x+1)=0 \Rightarrow x=-3, x=-1$ . (x-intercepts)  
 $f'(x)=0$  when  $-6x-12=-6(x+2)=0 \Rightarrow x=-2$  (CRIT. PT.)  
 $f''(x)=0$  when  $3x^2+12x+16=0 \Rightarrow x=\frac{-12 \pm \sqrt{144-192}}{6} \Rightarrow$  NEVER. (NO PTS OF INFLECTION)

Vertical asymptotes at  $x=0, x=-4$ .

$\lim_{x \rightarrow \pm\infty} f(x) = +1 \Rightarrow y=1$  a horizontal asymptote.



$f(-2) = \frac{-1}{-4} = \frac{1}{4}$



② a  $\lim_{x \rightarrow 1} \frac{x^4 - 5x^2 + 3x + 1}{x^3 - 2x + 1} \Rightarrow \frac{1-5+3+1}{1-2+1} = \frac{0}{0}$   
 $\therefore = \lim_{x \rightarrow 1} \frac{4x^3 - 10x + 3}{3x^2 - 2} = \frac{4-10+3}{3-2} = \boxed{-3}$

b  $\lim_{x \rightarrow 0} \frac{\sin x - x}{(\cos x - 1)x} \rightarrow \frac{0}{0} \therefore = \lim_{x \rightarrow 0} \frac{\cos x - 1}{\cos x - 1 - x \sin x} \rightarrow \frac{0}{0}$   
 $\therefore = \lim_{x \rightarrow 0} \frac{-\sin x}{-2 \sin x - x \cos x} \rightarrow \frac{0}{0} \therefore = \lim_{x \rightarrow 0} \frac{-\cos x}{-3 \cos x + x \sin x} = \frac{-1}{-3} = \boxed{\frac{1}{3}}$

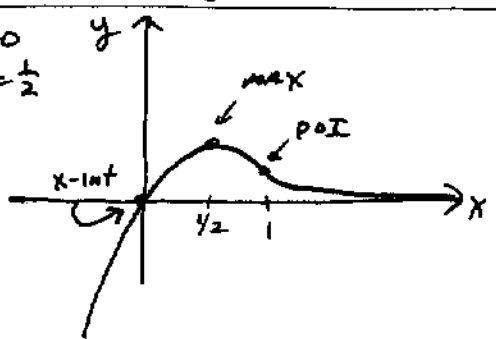
c LOOK AT  $\lim_{x \rightarrow \infty} \ln \left( \frac{x+4}{x-1} \right)^x = \lim_{x \rightarrow \infty} x \ln \left( \frac{x+4}{x-1} \right) = \lim_{x \rightarrow \infty} \frac{\ln(x+4) - \ln(x-1)}{1/x} \rightarrow \frac{0}{0}$   
 $\therefore = \lim_{x \rightarrow \infty} \frac{\frac{1}{x+4} - \frac{1}{x-1}}{(-1/x^2)} = \lim_{x \rightarrow \infty} -x^2 \left[ \frac{(x-1) - (x+4)}{(x+4)(x-1)} \right] = \lim_{x \rightarrow \infty} \frac{5x^2}{x^2+3x-4} = \boxed{5}$   
 $\therefore \ln[\text{LIMIT}] = 5, \text{ SO LIMIT} = \boxed{e^5}$

③ a  $f(x) = x^4 - 2, f'(x) = 4x^3, x_0 = 1, x_1 = N(x_0) = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{4} = \boxed{\frac{5}{4}}$

b  $f(x) = x^3 - 7x - 5, f'(x) = 3x^2 - 7, x_0 = 3, x_1 = 3 - \frac{f(3)}{f'(3)} = 2.95, x_2 = 2.95 - \frac{f(2.95)}{f'(2.95)} \approx 2.9428 \dots$

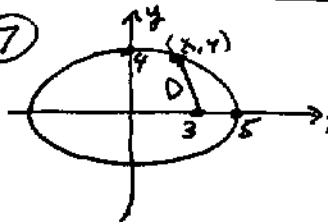
④  $f(x) = x e^{-2x}, f'(x) = 2x e^{-2x} + e^{-2x} = e^{-2x}(1+2x), f''(x) = -2e^{-2x} + (1+2x)e^{-2x}(-2) = e^{-2x}(4x-4)$   
 x-intercept at  $x=0$   
 CRIT PT AT  $x = \frac{1}{2}$   
 P.O.I at  $x=1$

x	0	1/2	1
f(x)	0	1/2e	1/e^2

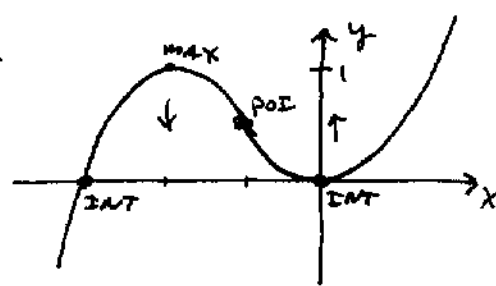


⑤  $g(0) = 5, g(2) = 11$  MVT says there's a  $c$  between 0 and 2 such that  
 $g'(c) = \frac{g(2) - g(0)}{2 - 0} = \frac{11 - 5}{2} = \frac{6}{2} = 3$ , i.e. slope of TL = 3.

⑥ Area  $A = 3 \cdot y + 2(\frac{1}{2}xy) = 3y + xy$  and  $x^2 + y^2 = 9$   
 $A = y(3+x) = (3+x)\sqrt{9-x^2} = A(x) \quad 0 \leq x \leq 3$   
 $A'(x) = (3+x) \frac{1}{2}(9-x^2)^{-1/2}(-2x) + (9-x^2)^{1/2} = \frac{-x(3+x) + (9-x^2)}{\sqrt{9-x^2}} = \frac{9-3x-2x^2}{\sqrt{9-x^2}}$   
 CRIT POINT when  $2x^2 + 3x - 9 = 0 \Rightarrow x = \frac{-3 \pm \sqrt{9+72}}{4} = \frac{-3 \pm \sqrt{81}}{4} = \frac{-3 \pm 9}{4}$   
 $\Rightarrow x = \frac{3}{2}$  (in domain).  
 $A(0) = 9 \quad A(\frac{3}{2}) = \frac{9}{2} \sqrt{9 - \frac{9}{4}} = \frac{9}{2} \sqrt{\frac{27}{4}} = \frac{27}{4} \sqrt{3} \approx 11.69$   
 $A(3) = 0$  So  $x = \frac{3}{2}$  gives MAXIMUM AREA.

⑦   $D^2 = (x-3)^2 + y^2$  and  $\frac{x^2}{25} + \frac{y^2}{16} = 1 \Rightarrow y^2 = 16(1 - \frac{x^2}{25})$   
 So  $D^2 = (x-3)^2 + 16 - \frac{16}{25}x^2 = F(x) \quad -5 \leq x \leq 5$   
 $F'(x) = 2(x-3) - \frac{32}{25}x = \frac{18}{25}x - 6 = 0 \Rightarrow x = 8\frac{1}{3}$  (OUT OF DOMAIN)  
 So MINIMUM DISTANCE AT ENDPOINTS  $\rightarrow (5, 0)$  clearly the closest.

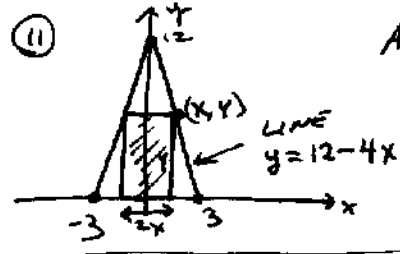
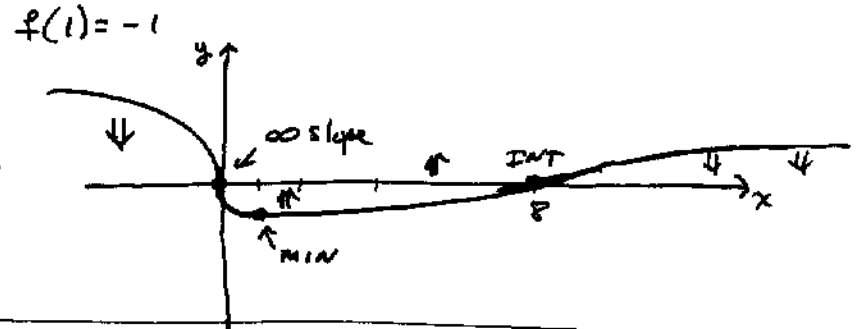
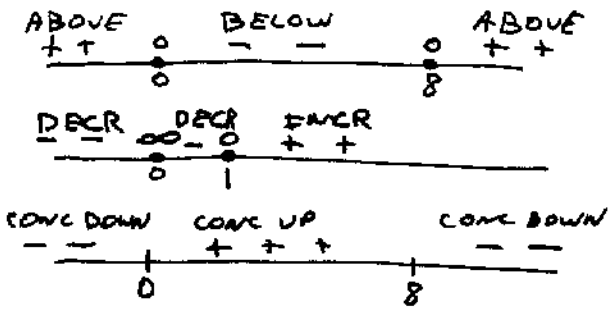
⑧  $f(x) = 3x^2 + 2x^3 = x^2(3+2x)$  x-intercept at  $x=0$  and  $x=-3/2$   
 $f'(x) = 6x + 6x^2 = 6x(1+x)$  CRIT PTS AT  $x=0, x=-1$   
 $f''(x) = 6 + 12x = 6(1+2x)$   $x = -1/2$  possible P.O.I.  
 BELOW  $\circ$  ABOVE  $\circ$  ABOVE  $\circ$   
 INCREASE  $\circ$  DECREASE  $\circ$  INCREASE  $\circ$   
 CONV. DOWN  $\circ$  CONV. UP  $\circ$   
 $-3/2 \quad -1 \quad -1/2$



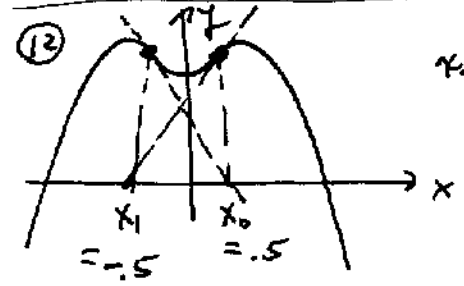
⑨  $a(t) = -t^2 \quad x(0) = 0 \quad v(0) = v_0 \quad S_{max} = 4 \Rightarrow v(t) = -\frac{t^3}{3} + C \Rightarrow v(t) = v_0 - \frac{1}{3}t^3$   
 $v(0) = C = v_0$   
 So  $s(t) = v_0 t - \frac{1}{12}t^4 + C_1$   
 $s(0) = 0 - 0 + C_1 = 0 \Rightarrow C_1 = 0 \Rightarrow s(t) = v_0 t - \frac{1}{12}t^4$   
 MAX DISTANCE when  $v=0 \Rightarrow v_0 - \frac{1}{3}t^3 = 0 \Rightarrow v_0 = \frac{t^3}{3}$   $(3v_0 = t^3)$   
 AT THAT TIME,  $S = 4$ , so  $4 = v_0 t - \frac{1}{12}t^4$   
 Solve SIMULTANEOUSLY  $\Rightarrow 4 = \frac{t^3}{3}t - \frac{1}{12}t^4 \Rightarrow 48 = 4t^4 - t^2$   
 $\Rightarrow 4t^4 - t^2 - 48 = 0$   
 $\Rightarrow t^2 = \frac{1 \pm \sqrt{1+768}}{8} \Rightarrow t^2 = \frac{1 + \sqrt{769}}{8} = 3.5913 \dots$   
 $\therefore t \approx 1.895 \Rightarrow v_0 = 2.2686 \text{ mi/hr.}$

⑩  $f(x) = x^{2/3} - 2x^{1/3} = x^{1/3}(x^{1/3} - 2) = 0$  when  $x=0, x=8$   
 $f'(x) = \frac{2}{3}x^{-1/3} - \frac{2}{3}x^{-2/3} = \frac{2}{3} \frac{(x^{1/3} - 1)}{x^{2/3}} = 0$  when  $x=1$  vert slope when  $x=0$

$f''(x) = -\frac{2}{9}x^{-4/3} + \frac{4}{9}x^{-5/3} = \frac{2}{9} \frac{(-x^{1/3} + 2)}{x^{5/3}} = 0$  when  $x=8$ , possible POI.  
 $x=0$  ALSO possible POI



Area  $A = (2x)y = 2x(12-4x) = 24x - 8x^2 = A(x)$   
 $0 \leq x \leq 3$        $A'(x) = 24 - 16x = 0 \Rightarrow x = 3/2$   
 $A(0) = A(3) = 0$        $\therefore$  MAX AREA when  $x = 3/2$   
 $A(3/2) = 36 - 8(\frac{9}{4}) = 36 - 18 = 18 = A_{max}$   
 $A_{max} = 18 \text{ in}^2$



$x_0 = .5 \rightarrow x_1 = -.5 \rightarrow x_2 = .5 \rightarrow x_3 = -.5$  etc.  
 $.5 \rightarrow -.5 \rightarrow +.5 \rightarrow -.5 \rightarrow \dots$   
 CYCLES BACK AND FORTH. BAD INITIAL GUESS.  
 Better initial guesses give roots AT  
 $x = \pm 1.36676 \dots$

⑬ Let  $x = \#$  doctors hired       $10000x + 4000y = 100000 \Rightarrow 5x + 2y = 50$   
 $y = \#$  nurses hired       $N = kxy = kx(\frac{50-5x}{2}) = k(25x - \frac{5}{2}x^2)$   
 $0 \leq x \leq 10$        $N'(x) = k(25 - 5x) = 0 \Rightarrow x = 5$   
 $N(0) = 0$        $N(10) = 0$        $N(5) = k(125 - \frac{125}{2}) = 125k/2$   
 $x = 5 \text{ DOCTORS}$  AND  $y = \frac{25}{2} = 12.5 \text{ NURSES}$ .



$V = \pi r^2 h = 40$       Surf Area  $A = 2\pi r^2 + 2\pi r h$   
 $h = \frac{40}{\pi r^2}$        $= 2\pi r^2 + 2\pi r(\frac{40}{\pi r^2}) = 2\pi r^2 + \frac{80}{r} = A(r)$   
 $A'(r) = 4\pi r - \frac{80}{r^2} = 0 \Rightarrow 4\pi r = \frac{80}{r^2} \Rightarrow r^3 = \frac{20}{\pi}$

So  $r = \sqrt[3]{\frac{20}{\pi}} = 1.853 \text{ cm}$   
 $h = \frac{40}{\pi r^2} = 3.707 \text{ cm}$

Note  $\frac{h}{r} = \frac{40}{\pi r^2} \cdot \frac{1}{r} = \frac{40}{\pi r^3} = \frac{40}{\pi \cdot \frac{20}{\pi}} = 2$   
 $\Rightarrow$  (height = twice the radius)

⑮ Let  $x = \text{no. of } 5\% \text{ reductions}$   
 price =  $100000(1 - .05x)$   
 people =  $60 + 10x$

$$\Rightarrow \text{Revenue} = [100000(1 - .05x)] [60 + 10x]$$

$$= (100000 - 50000x)(60 + 10x)$$

$$\Rightarrow R(x) = 6000000 + 700000x - 50000x^2$$

$R'(x) = 700000 - 100000x = 0 \Rightarrow x = 7 \Rightarrow \underline{35\% \text{ reduction}}$ .  
 However, we must also consider the capacity of the plane!

$x = 7 \Rightarrow \text{people} = 60 + 70 = 130 \Rightarrow \text{overcapacity}$ .

$x = 6 \Rightarrow 120 \text{ people is endpoint}$ .  $R(0) = 6 \text{ million FF}$  MAXIMUM  
 $R(6) = 8.4 \text{ million FF} \leftarrow \text{REVENUE}$

⑯  $v(t) = t^2 = \frac{dx}{dt}$   $x(t) = \frac{1}{3}t^3 + C$

$$x(2) - x(1) = \left(\frac{8}{3} + C\right) - \left(\frac{1}{3} + C\right) = \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3}}$$

⑰  $a(t) = -5 - 4t \Rightarrow v(t) = -5t - 2t^2 + C$   $x(t) = 25t - \frac{5}{2}t^2 - \frac{2}{3}t^3 + C_1$   
 $v(0) = 25 \text{ ft/sec}$   $v(0) = 0 + C = 25 \Rightarrow C = 25$   $\Rightarrow x(0) = C_1 = 0$   
 $x(0) = 0 \text{ ft}$   $\text{So } v(t) = 25 - 5t - 2t^2$   $\text{So } x(t) = 25t - \frac{5}{2}t^2 - \frac{2}{3}t^3$

⑱ BAGEL NOT MOVING when  $v = 0 \Rightarrow 2t^2 + 5t - 25 = 0$   
 $t = \frac{-5 \pm \sqrt{25 + 200}}{4} = \frac{-5 \pm 15}{4} \Rightarrow t = \boxed{\frac{5}{2} \text{ sec}}$

$x(0) = 0$   
 $x\left(\frac{5}{2}\right) = \frac{125}{2} - \frac{5}{2} \cdot \frac{25}{4} - \frac{2}{3} \cdot \frac{125}{8} = \frac{875}{24} = 36 \frac{1}{24} \text{ feet (outside } [0, 2])$   
 $x(2) = 50 - 10 - \frac{16}{3} = \boxed{34 \frac{2}{3} \text{ feet} = \text{DISTANCE TRAVELLED}}$

⑲ a)  $f'(x) = \frac{1}{1 + \left(\frac{x-1}{x+1}\right)^2} \cdot \frac{(x+1) - (x-1)}{(x+1)^2} = \frac{2}{(x+1)^2 + (x-1)^2} = \frac{2}{x^2 + 2x + 1 + x^2 - 2x + 1} = \frac{1}{x^2 + 1}$

Note:  $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{x^2 + 1}$  also, so  $\tan^{-1}\left(\frac{x-1}{x+1}\right) - \tan^{-1}(x) = \text{constant}$

⑳  $h'(x) = \frac{(x^3 + 2) \left[ x^2 \cdot \left(\frac{3}{3x+4}\right) + 2x \ln(3x+4) \right] - [x^2 \ln(3x+4)](3x^2)}{(x^3 + 2)^2}$  [note:  $\ln(1) = 0$ ]

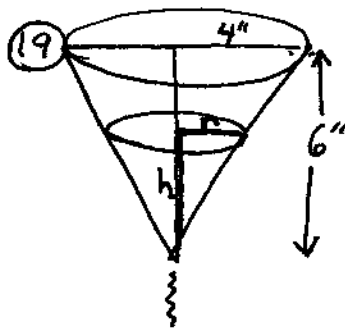
$$h'(-1) = \frac{(1) [1 \cdot 3 - 2 \ln(1)] - [1 \cdot \ln(1)](3)}{1} = \boxed{3}$$

㉑  $g'(t) = 3\pi t^2 + (3^{\pi t} \ln 3) \cdot \pi + \pi^3$

㉒  $e^{y^2 + y} = xy + 1$  DIFF. IMPLICITLY

$$(e^{y^2 + y})(2y + 1) \frac{dy}{dx} = x \cdot \frac{dy}{dx} + y$$

plug in  $x=0$   
 $y=-1 \Rightarrow -\frac{dy}{dx} = -1 \Rightarrow \boxed{\frac{dy}{dx} = 1}$



SIMILAR TRIANGLES  $\frac{r}{h} = \frac{4}{6} \Rightarrow 6r = 4h \Rightarrow h = \frac{3}{2}r$

$$V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi r^2 \left(\frac{3}{2}r\right) = \frac{\pi}{2} r^3 = V$$

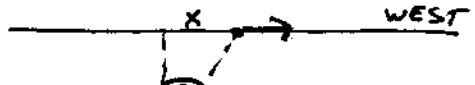
Take derivative w.r.t. time  $t \Rightarrow \frac{3\pi}{2} r^2 \frac{dr}{dt} = \frac{dV}{dt}$

We know that  $\frac{dV}{dt} = -2 \text{ in}^3/\text{min}$ . When  $h = 3 \text{ in}$ ,  $r = 2 \text{ in}$ .

$$\text{So } \frac{3\pi}{2} \cdot 4 \frac{dr}{dt} = -2 \Rightarrow 6\pi \frac{dr}{dt} = -2 \Rightarrow \frac{dr}{dt} = -\frac{1}{3\pi} \text{ in/sec}$$

So  $\frac{dh}{dt} = \frac{3}{2} \frac{dr}{dt} = -\frac{1}{2\pi} \text{ in/sec}$

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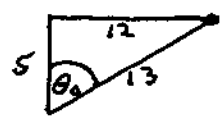


WEST  $\frac{dx}{dt} = 507 \text{ mi/hr}$

use this angle. other choices OK

$$\frac{x}{5} = \tan \theta \Rightarrow \frac{1}{5} \frac{dx}{dt} = \sec^2 \theta \frac{d\theta}{dt}$$

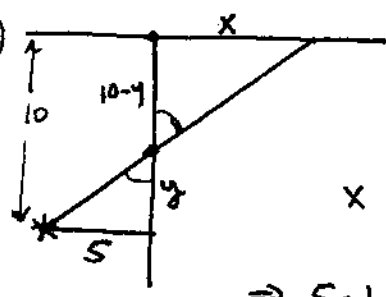
When  $x = 12 \text{ mi}$ ,  $\sec \theta = \frac{13}{5}$



$$\text{So } \frac{1}{5} (507) = \left(\frac{13}{5}\right)^2 \frac{d\theta}{dt} = \frac{169}{25} \frac{d\theta}{dt}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{507}{5} \cdot \frac{25}{169} = 5.3 = \boxed{15 \text{ rad/hr}}$$

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Similar Triangles  $\frac{x}{10-y} = \frac{5}{y}$

$$\Rightarrow xy = 50 - 5y$$

$$\frac{dy}{dt} = 1 \text{ m/s}$$

$$x \frac{dy}{dt} + y \frac{dx}{dt} = -5 \frac{dy}{dt}$$

PLUG IN  $\frac{dy}{dt} = 1 \text{ m/s}$

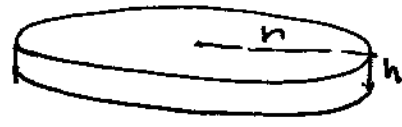
$$y = 5, x = 5$$

$$\Rightarrow 5 \cdot 1 + 5 \cdot \frac{dx}{dt} = -5 \cdot 1$$

$$\Rightarrow 5 \frac{dx}{dt} = -10 \Rightarrow \frac{dx}{dt} = -2 \text{ m/s}$$

(2 m/s to left)

22



$$V = \pi r^2 h \quad \frac{dV}{dt} = -5 \text{ ft}^3/\text{hr}$$

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi r h \frac{dr}{dt}$$

$$r = 500 \text{ ft}, h = .01 \text{ ft}, \frac{dh}{dt} = -.001 \text{ ft/hr}$$

$$\Rightarrow -5 = \pi(250000)(-.001) + 2\pi(500)(.01) \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{250\pi - 5}{10\pi} \text{ ft/hr}$$

(RADIUS INCREASING)

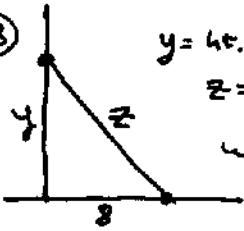
23

$$V = A \cdot h \quad \frac{dV}{dt} = A \frac{dh}{dt} + h \frac{dA}{dt} \Rightarrow -5 = \pi(250000)(-.001) + (.01) \frac{dA}{dt}$$

$$\text{So } \frac{dA}{dt} = \frac{250\pi - 5}{.01} = (25000\pi - 500) \text{ ft}^2/\text{hr} \quad \text{INCREASING}$$

Note! Could also use  $A = \pi r^2 \quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 1000\pi \frac{dr}{dt}$  to get this.

23



$y =$  ht. of monkey above ground  
 $z =$  length of rope  
 weight of monkey is irrelevant

$$64 + y^2 = z^2$$

$$2y \frac{dy}{dt} = 2z \frac{dz}{dt}$$

$$12 \cdot 4 = 20 \frac{dz}{dt}$$

$$\Rightarrow \frac{dz}{dt} = \boxed{2.4 \text{ ft/sec}}$$

$$y = 6' \Rightarrow z = 10'$$

$$\frac{dy}{dt} = 4 \text{ ft/sec}$$

24

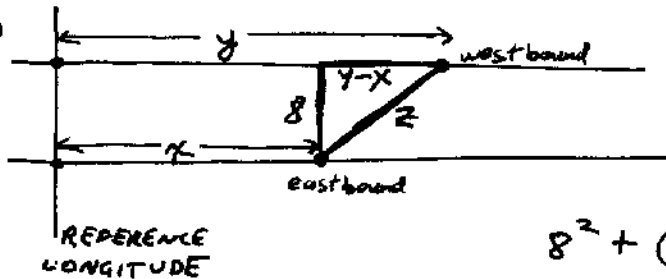


$$V = \frac{4}{3} \pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\text{If } \frac{dr}{dt} = 2 \text{ in/min, } r = 5 \text{ in, then } \frac{dV}{dt} = 4\pi \cdot 25 \cdot 2$$

$$= \boxed{200\pi \text{ in}^3/\text{min}}$$

25



$$\frac{dy}{dt} = -20 \text{ naut mi/hr}$$

$$\frac{dx}{dt} = +20 \text{ naut mi/hr}$$

$$8^2 + (y-x)^2 = z^2$$

$$64 + (y-x)^2 = z^2$$

$$\text{Differentiate w.r.t time } \Rightarrow 2(y-x) \left( \frac{dy}{dt} - \frac{dx}{dt} \right) = 2z \frac{dz}{dt}$$

$$\text{When ships are 10 naut mi apart, } z = 10 \Rightarrow (y-x) = 6 \text{ mi}$$

$$\cdot \text{ So } 2(6)(-20 - 20) = 2 \cdot 10 \frac{dz}{dt}$$

$$\Rightarrow 12(-40) = 20 \frac{dz}{dt}$$

$$\Rightarrow \frac{dz}{dt} = -\frac{480}{20} = \boxed{-24 \text{ naut mi/hr}}$$