SOLUTIONS TO EXAM 1 REVIEW PROBLEMS – MATH 1a – Fall 1999

① (a) $g(t) = \tan t \cdot (2+t^3)^5$

$$g'(t) = \tan t \cdot 5(2+t^3)^4(3t^2) + \sec^2 t \cdot (2+t^3)^5$$
$$= (2+t^3)^4\left[15t^2 \tan t + (2+t^3)\sec^2 t\right]$$

(b) $f(x) = \tan\left(\sqrt{\frac{x-1}{x+1}}\right) = \tan\left[\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}}\right]$

$$f'(x) = \sec^2\left(\sqrt{\frac{x-1}{x+1}}\right) \cdot \frac{1}{2}\left(\frac{x-1}{x+1}\right)^{-\frac{1}{2}}\left(\frac{(x+1)\cdot 1 - (x-1)\cdot 1}{(x+1)^2}\right)$$
$$= \sec^2\left(\sqrt{\frac{x-1}{x+1}}\right) \cdot \frac{1}{2}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}}\left(\frac{2}{(x+1)^2}\right) = \sec^2\left(\sqrt{\frac{x-1}{x+1}}\right) \cdot \frac{1}{(x-1)^{1/2}(x+1)^{3/2}}$$

(c) $f(x) = \cos^2\left(\frac{3}{x}\right) = \left[\cos\left(\frac{3}{x}\right)\right]^2$

$$f'(x) = 2\left[\cos\left(\frac{3}{x}\right)\right]^1\left[-\sin\left(\frac{3}{x}\right)\right]\left[-\frac{3}{x^2}\right] = \frac{6}{x^2}\cos\left(\frac{3}{x}\right)\sin\left(\frac{3}{x}\right)$$

(d) $y = \cos^4(x^5) = \left[\cos(x^5)\right]^4$

$$\frac{dy}{dx} = 4\left[\cos(x^5)\right]^3\left[-\sin(x^5)\right](5x^4) = -20x^4\cos^3(x^5)\sin(x^5)$$

(e) $f(x) = \sin^2 3x + \cos^2 3x = 1$ (by Pythagorean identity).
Therefore $f'(x) = 0$.

(f) $f(x) = \frac{3}{\sqrt[3]{x}} - x - \frac{1}{x} = 3x^{-\frac{1}{3}} - x - x^{-1}$

$$f'(x) = -x^{-4/3} - 1 + x^{-2} = -\frac{1}{x^{4/3}} - 1 + \frac{1}{x^2}$$
$$f'(1) = -\frac{1}{1} - 1 + \frac{1}{1} = -1 - 1 + 1 = \boxed{-1}.$$

(g) $x^2 + xy + 2y = 1$ Think $y = y(x)$ and use Chain Rule.

$$2x + x\frac{dy}{dx} + y + 2\frac{dy}{dx} = 0$$
$$(x+2)\frac{dy}{dx} = -2x - y \implies \frac{dy}{dx} = \frac{-2x-y}{x+2}$$

(2)

x	0.2	0.4	0.6	0.8	1.0
$f(x)$	20	25	27	21	15

There are a number of reasonable ways to answer this one:

(a) FOR $f'(0.6)$, look to either side. With $\Delta x = 0.2 \Rightarrow \dfrac{f(0.8)-f(0.6)}{.2} = -\dfrac{6}{.2} = -30$

TO left, with $\Delta x = -0.2 \Rightarrow \dfrac{f(0.4)-f(0.6)}{-.2} = \dfrac{-2}{-.2} = 10$

A really good estimate isn't really possible, but the average of the two should be reasonably close $\Rightarrow$ $\boxed{f'(0.6) \cong -10}$

(b) FOR $f''(0.6)$, MAKE A TABLE OF ESTIMATES for $f'(x)$. LOOKING AT MIDPOINTS, we have the following estimates for the derivatives:

x	0.3	0.5	0.7	0.9
$f'(x)$	25	10	-30	-30

FOR example, $f'(0.3) \cong \dfrac{f(.4)-f(.2)}{.2} = 25$

A reasonable estimate for $f''(0.6)$ is $\dfrac{f'(0.7)-f'(0.5)}{.2} = \dfrac{-30-10}{.2}$

So $f''(0.6) \cong -200$

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(3) $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

For $x \neq 0$, $f'(x) = x^2 \cdot \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) + 2x \sin\left(\frac{1}{x}\right)$

$= -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right)$

(note that this is not defined for $x = 0$.)

This function is, in fact continuous at $x = 0$.

$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq +x^2$ (since $\sin\left(\frac{1}{x}\right)$ bounded between ± 1)

Since $x^2 \to 0$ as $x \to 0$, $\lim_{x\to 0}\left[x^2 \sin\left(\frac{1}{x}\right)\right] = 0$ by Squeeze theorem.

So $\lim_{x\to 0} f(x) = f(0)$, and f is therefore continuous at $x = 0$.

For $f'(0)$, calculate the difference quotient $f'(0) = \lim_{x\to 0}\left[\dfrac{f(x)-f(0)}{x-0}\right]$

$= \lim_{x\to 0}\left[\dfrac{x^2 \sin\left(\frac{1}{x}\right) - 0}{x - 0}\right] = \lim_{x\to 0}\left[x \sin\left(\frac{1}{x}\right)\right]$.

We have that $-x \leq x \cdot \sin\frac{1}{x} \leq +x$, and $x \to 0$

$\Rightarrow$ Again by Squeeze theorem $\lim_{x\to 0}\left(x \sin\frac{1}{x}\right) = 0$.

So $f'(0) = 0$.

(4) The only important thing is that the limit represent the slope of the tangent line at $x = a$.

(a), (c), (d), and (e) all do this.

In (a), we're just using $2h$ instead of h, but idea is the same.

In (c), we're taking points to either side, the squeezing together.

(5) (a) At $x=0$, $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{(x-0)} = \lim\limits_{x\to 0} \dfrac{\frac{1}{2x+1} - 1}{x}$

$$= \lim_{x\to 0} \frac{1}{x}\left[\frac{1-(2x+1)}{2x+1}\right] = \lim_{x\to 0} \frac{-2x}{x(2x+1)} = \lim_{x\to 0} \frac{-2}{2x+1} = \boxed{-2}$$

If you find $f'(x)$ for any x, then $f'(x) = \lim\limits_{h\to 0}\left[\dfrac{f(x+h)-f(x)}{h}\right]$

$$= \lim_{h\to 0}\left[\frac{\frac{1}{2(x+h)+1} - \frac{1}{2x+1}}{h}\right] = \lim_{h\to 0}\frac{1}{h}\left[\frac{(2x+1)-(2x+2h+1)}{(2x+h+1)(2x+1)}\right]$$

$$= \lim_{h\to 0}\frac{-2h}{h(2x+h+1)(2x+1)} = \lim_{h\to 0}\frac{-2}{(2x+h+1)(2x+1)} = \frac{-2}{(2x+1)^2}.$$

then plug in $x=0$ to get $f'(0) = -2$.

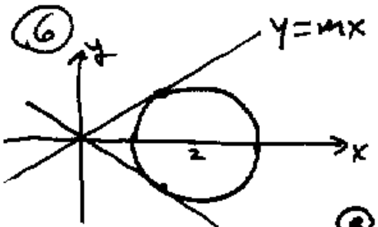
(b) $f(x) = \dfrac{1}{x^2}$ $\quad f(x+h) = \dfrac{1}{(x+h)^2}$

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \frac{x^2-(x+h)^2}{h\,(x+h)^2\cdot x^2}$$

$$= \frac{x^2-(x^2+2hx+h^2)}{h(x+h)^2\,x^2} = \frac{-2hx-h^2}{h(x+h)^2x^2} = \frac{-2x-h}{(x+h)^2x^2}.$$

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] = \lim_{h\to 0}\frac{-2x-h}{(x+h)^2x^2} = \frac{-2x}{x^2\cdot x^2} = -\frac{2}{x^3}$$

(6)



(a) $\boxed{x^2-4x+y^2+3=0}$

$\Downarrow$

$(x-2)^2+y^2=1$

CURVE IS A CIRCLE, but it's not necessary to know this in order to solve.

Line through origin given by $\boxed{y=mx}$ (b)

Find slope by Implicit differentiation:

$2x-4+2y\dfrac{dy}{dx}=0 \Rightarrow \dfrac{dy}{dx} = \dfrac{4-2x}{2y} = \boxed{\dfrac{2-x}{y}=m}$ (c)

(a)+(b) $\Rightarrow x^2-4x+m^2x^2+3=0 \Rightarrow (1+m^2)x^2-4x+3=0$

(b)+(c) $\Rightarrow 2-x = my = m^2x^2 \Rightarrow (1+m^2)x = 2 \quad \Rightarrow$

(6 cont)

Substituting $(1+m^2)x=2$ into $(1+m^2)x\cdot x-4x+3=0$

gives $2x-4x+3=0 \Rightarrow 2x=3 \Rightarrow \boxed{x=\frac{3}{2}}$

Therefore $(1+m^2)\frac{3}{2}=2 \Rightarrow 1+m^2=\frac{4}{3} \Rightarrow m^2=\frac{1}{3}$ $\boxed{m=\frac{\pm 1}{\sqrt{3}}}$

therefore the two lines are $y=\frac{1}{\sqrt{3}}x$ and $y=-\frac{1}{\sqrt{3}}x$.

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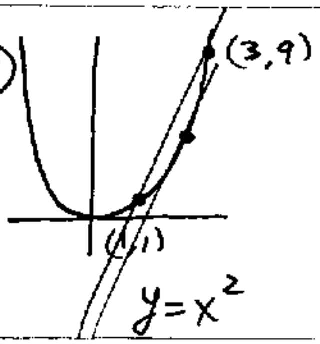
7 

SLOPE OF SECANT LINE $=\frac{9-1}{3-1}=\frac{8}{2}=4$.

SLOPE OF TL given by derivative $\frac{dy}{dx}=2x$.

want $2x=4 \Rightarrow x=2$

$\Rightarrow$ POINT $(2,4)$ on parabola.

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8 $s(t)=\frac{2}{3}t^3-7t^2+20t+8$

(a) velocity $=v(t)=s'(t)=2t^2-14t+20=2(t^2-7t+10)$
$=\boxed{2(t-2)(t-5)}$

(b) acceleration $=v'(t)=\boxed{4t-14}$

(c) moves to right when $v(t)>0$

$$\begin{array}{ccccccc} & -- & & +- & & ++ & \\ & \oplus & & \ominus & & \oplus & \\ 0 & & 2 & & 5 & & t \end{array}$$

$v(t)>0$ when $\boxed{0\le t<2 \text{ and when } t>5}$.

(d) velocity decreasing when $v'(t)=a(t)<0 \Rightarrow 4t-14<0$
$\Rightarrow \boxed{0\le t<\frac{7}{2}}$

(e) OBJECT CHANGES DIRECTION when velocity changes sign $\Rightarrow \boxed{t=2,\ t=5}$

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9 $\frac{d}{dx}[f(\sec x)]=f'(\sec x)\cdot\frac{d}{dx}(\sec x)=f'(\sec x)\cdot\sec x\tan x$.

But $f'(x)=(x^2-1)^{-\frac{1}{2}}=\frac{1}{\sqrt{x^2-1}}$. So $f'(\sec x)=\frac{1}{\sqrt{\sec^2 x-1}}$.

SINCE $\tan^2 x+1=\sec^2 x \Rightarrow \sec^2 x-1=\tan^2 x$.

So $f'(\sec x)=\frac{1}{\sqrt{\tan^2 x}}=\frac{1}{|\tan x|}=\frac{1}{\tan x}$ for $x\in(0,\frac{\pi}{2})$.

Therefore $\frac{d}{dx}[f(\sec x)]=\frac{1}{\tan x}\cdot\sec x\tan x=\underline{\underline{\sec x}}$.

(10) FIRST, CHECK THAT $(0,-1)$ SATISFIES $\sec(y^2+y) = xy+1$

$$\Rightarrow \sec(1-1) \stackrel{?}{=} 0+1$$

$$\sec(0) = 1 \quad \checkmark$$

Next, find $\frac{dy}{dx}$ via implicit differentiation:

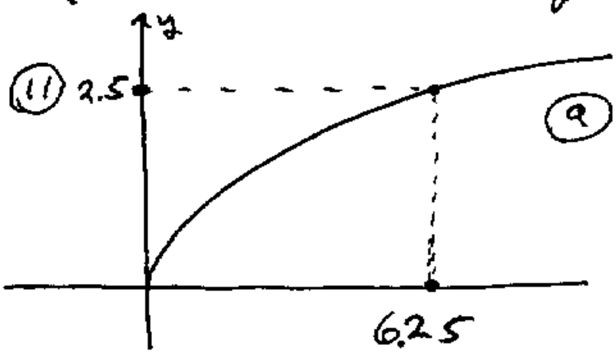
$$\left[\sec(y^2+y)\tan(y^2+y)\right]\cdot\left[(2y+1)\frac{dy}{dx}\right] = x\frac{dy}{dx} + y$$

$$\text{At } (x,y) = (0,-1) \Rightarrow \left[\sec(0)\tan(0)\right](-1)\frac{dy}{dx} = 0-1$$

$$\Rightarrow \quad 0 = -1 \quad \Rightarrow \boxed{\frac{dy}{dx} \text{ does not exist at this point.}}$$

(You can check this also by treating $x = x(y) \Rightarrow \frac{dx}{dy} = 0$)

(11)



(a) $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

LINEAR APPROX. IS GIVEN BY

$$f(x) \cong f(a) + f'(a)(x-a)$$

for x near a

In this case, choose $a = 6.25$

then $f(6.25) = 2.5$ and $f'(6.25) = \frac{1}{2\cdot 2.5} = \frac{1}{5}$

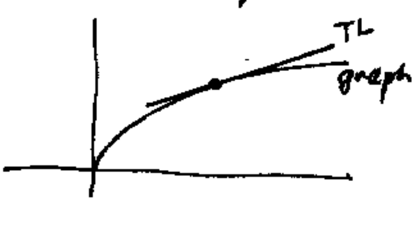
$$\sqrt{x} \cong f(6.25) + f'(6.25)(x-6.25)$$

$$\sqrt{x} = 2.5 + \frac{1}{5}(x-6.25) \quad \text{for } x \text{ near } 6.25$$

$$\text{So } \sqrt{6} \cong 2.5 + \frac{1}{5}(6-6.25) = 2.5 - \frac{.25}{5}$$

$$= 2.5 - .05 = \underline{\underline{2.45}}$$

(b) LINEAR APPROXIMATION IS GIVEN by TANGENT LINE TO GRAPH. FOR THIS FUNCTION, this tangent LINE LIES ABOVE THE GRAPH, SO THE APPROXIMATION WILL BE SLIGHTLY GREATER THAN THE ACTUAL VALUE of $\sqrt{6}$.



(Actual value $\sqrt{6} = 2.4494897\cdots\cdots$)