MATH 1a MIDTERM EXAM 2 REVIEW PROBLEMS — SOLUTIONS

1) a) $\lim_{x\to 0} \frac{\sin x - x}{(\cos x - 1)x} \to \frac{0}{0}$ $\therefore$ by L'Hôpital's Rule

$= \lim_{x\to 0} \frac{\cos x - 1}{(\cos x - 1) + (-\sin x)x} \to \frac{0}{0}$

$\therefore = \lim_{x\to 0} \frac{-\sin x}{-\sin x - x\cos x - \sin x}$

$= \lim_{x\to 0}\left(\frac{+\sin x}{2\sin x + x\cos x}\right) \to \frac{0}{0}$ $\therefore = \lim_{x\to 0} \frac{\cos x}{3\cos x - x\sin x} = \boxed{\frac{1}{3}}$

b) $\lim_{x\to\infty}\left(\frac{x+4}{x-1}\right)^x = L$ $\quad \ln L = \ln\left[\lim_{x\to\infty}\left(\frac{x+4}{x-1}\right)^x\right] = \lim_{x\to\infty}\left[\ln\left(\frac{x+4}{x-1}\right)^x\right]$

$= \lim_{x\to\infty}\left[x\cdot\ln\left(\frac{x+4}{x-1}\right)\right] \to \infty\cdot 0$

$= \lim_{x\to\infty}\left[\frac{\ln\left(\frac{x+4}{x-1}\right)}{\left(\frac{1}{x}\right)}\right] \to \frac{0}{0}$

$\therefore = \lim_{x\to\infty}\left[\frac{\left(\frac{x-1}{x+4}\right)\left(\frac{(x-1)-(x+4)}{(x-1)^2}\right)}{-\frac{1}{x^2}}\right] = \lim_{x\to\infty}\left[\frac{\left(\frac{x-1}{x+4}\right)\frac{-5}{(x-1)^2}}{-\frac{1}{x^2}}\right] = \lim_{x\to\infty}\frac{\frac{+5}{(x+4)(x-1)}}{+\frac{1}{x^2}}$

$= \lim_{x\to\infty}\frac{5x^2}{(x+4)(x-1)} = \lim_{x\to\infty}\left(\frac{5x^2}{x^2+3x-4}\right) \to \frac{\infty}{\infty}$

$\therefore = \lim_{x\to\infty}\left(\frac{10x}{2x+3}\right) \to \frac{\infty}{\infty}$ $\quad\therefore = \lim_{x\to\infty}\left(\frac{10}{2}\right) = 5$

So $\ln L = 5 \Rightarrow L = \boxed{e^5}$

2) $f(x) = x^3 - 7x - 5$ $\quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{1}{20} = \frac{59}{20}$

$f'(x) = 3x^2 - 7$

$x_0 = 3$

$f(3) = 27 - 21 - 5 = 1$ $\quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{59}{20} - \frac{\left(\frac{59}{20}\right)^3 - 7\left(\frac{59}{20}\right) - 5}{3\left(\frac{59}{20}\right)^2 - 7}$

$f'(3) = 20$

$f\left(\frac{59}{20}\right) = \left(\frac{59}{20}\right)^3 - 7\left(\frac{59}{20}\right) - 5$ $\quad = \frac{59}{20} - \frac{(59)^3 - 7\cdot 59\cdot 20^2 - 5\cdot 20^3}{3(59)^2\cdot 20 - 7\cdot 20^3}$

$f'\left(\frac{59}{20}\right) = 3\left(\frac{59}{20}\right)^2 - 7$ $\quad = \frac{59}{20} - \frac{179}{152860} = \boxed{\frac{450758}{152860}}$

$= 2.9488\cdots$

[Clearly, without the use of a calculator, it's generally not so easy to calculate too many iterates with Newton's method.]

3) $f(x) = 3x^2 + 2x^3$ x-Intercepts: $x^2(3+2x)=0 \Rightarrow x=0,\ x=-\frac{3}{2}$

$f'(x) = 6x + 6x^2$ CRIT PTS: $6x(1+x)=0 \Rightarrow x=0,\ x=-1$

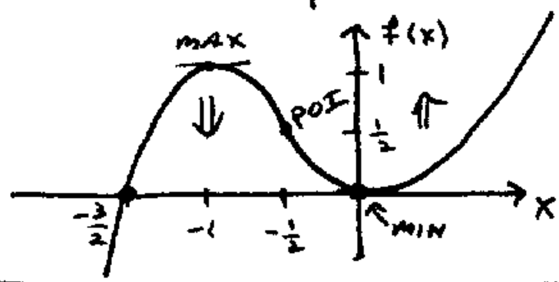
$f''(x) = 6 + 12x$ P.O.I: $6(1+2x)=0 \Rightarrow x=-\frac{1}{2}$

f: below $- -$ 0 ($-3/2$) above $+ +$ 0 (0) above $+ +$

f': INCR. $+ + +$ 0 (-1) DECR. $- -$ 0 (0) INCR. $+ +$

f'': CONC. DOWN $- - -$ 0 ($-\frac{1}{2}$) CONC. UP $+ +$

x	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0
$f(x)$	0	1	$\frac{1}{2}$	0



4) a) $f(x) = \frac{x-1}{x^2-2x} = \frac{x-1}{x(x-2)}$ x-intercept at $x=1$

vert. Asymptotes at $x=0,\ x=2$

$f'(x) = \frac{-x^2+2x-2}{x^2(x-2)^2} \Rightarrow$ NO CRITICAL POINTS

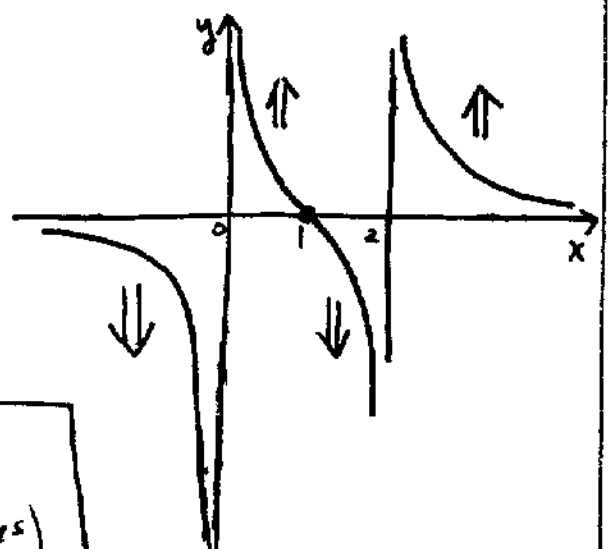
$f''(x) = \frac{2(x^3-3x^2+6x-4)}{x^3(x-2)^3} = \frac{2(x-1)(x^2-2x+4)}{x^3(x-2)^3} \Rightarrow$ P.O.I at $x=1$.

HORIZ. Asymptotes? $\lim_{x\to+\infty}\left(\frac{x-1}{x^2-2x}\right)=0$ and $\lim_{x\to-\infty}\left(\frac{x-1}{x^2-2x}\right)=0$

f: $- -$ VA (0) $+$ 0 (1) $-$ VA (2) $+ +$

f': $- -$ VA (0) $- -$ (1) VA (2) $- -$ Always Decreasing

f'': $-$ VA (0) $+$ 0 (1) $-$ VA (2) $+$



b) $f(x) = \frac{x^2-2x}{x-1} = (x-1) - \frac{1}{x-1}$ x-intercepts $x=0$ $x=2$

$f'(x) = 1 + \frac{1}{(x-1)^2}$ NO CRIT. PTS. (Always INCR.)

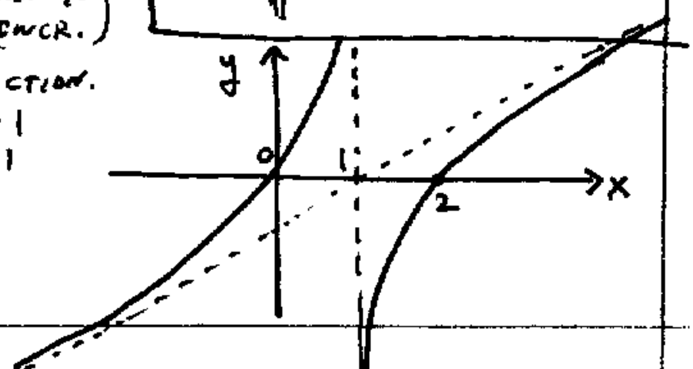
$f''(x) = \frac{-2}{(x-1)^3}$ NO PTS. OF INFLECTION.

CONC. DOWN FOR $x>1$

CONC. UP FOR $x<1$

$\boxed{\text{OBLIQUE ASYMPTOTE: } y = x-1}$

$\boxed{\text{vertical Asymptote: } x=1}$



5) $y = x^{\frac{2}{3}} - 2x^{\frac{1}{3}} = x^{\frac{1}{3}}\left(x^{\frac{1}{3}} - 2\right)$

$\boxed{\text{X-intercepts at } x=0,\ x=8}$
(Since $\sqrt[3]{x} = 2 \Rightarrow x = 8$)

$$\frac{dy}{dx} = \frac{2(x^{1/3}-1)}{3x^{2/3}}$$

$\boxed{\text{CRIT PTS AT } x=1 \text{ SLOPE} = 0;\quad x=0 \text{ Infinite SLOPE}}$

$$\frac{d^2y}{dx^2} = \frac{-2(x^{1/3}-2)}{9x^{5/3}}$$

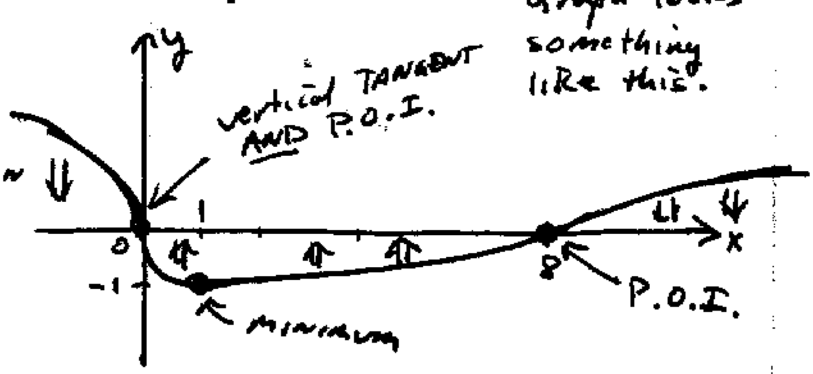
$\boxed{\text{PTS of INFLECTION AT } x=8 \text{ and possibly at } x=0.}$

y: ABOVE $+$ 0 BELOW $- - -$ 0 ABOVE $+ +$ (marks at 0, 1, 8)

$\frac{dy}{dx}$: DEC $-$ ∞ DEC $-$ INCREASING $+ + + + +$ (marks at 0, 1)

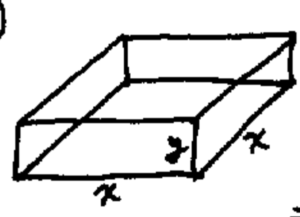
$\frac{d^2y}{dx^2}$: $-$ CONC. UP $+ + +$ CONC. DOWN $- -$ (marks at 0, 1, 8)

x	0	1	8
y	0	-1	0



Graph looks something like this.

6)



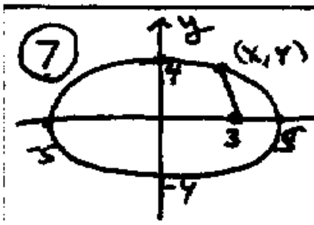
$V = x^2 y$ is to be MAXIMIZED.
$A = 4xy + x^2 = 12$ is surface area of the box.
USE THIS TO ELIMINATE $y = \frac{12 - x^2}{4x}$

$$\Rightarrow V(x) = x^2\left(\frac{12-x^2}{4x}\right) = 3x - \frac{1}{4}x^3$$

BOUNDS ARE THAT $x \geq 0$ OR THAT height must be $\geq 0 \Rightarrow x \leq \sqrt{12}$
So $0 \leq x \leq \sqrt{12}$. NOW FIND CRITICAL POINTS:
$V'(x) = 3 - \frac{3}{4}x^2 = 0 \Rightarrow x^2 = 4 \Rightarrow \boxed{x = 2}\ \boxed{y = 1}$

ENDPOINTS GIVE $V=0$. $V(2) = 6 - 2 = \boxed{4\ ft^3 = V_{max}}$

7)



PICK (x,y) on ELLIPSE. So $\frac{x^2}{25} + \frac{y^2}{16} = 1$ $\quad y^2 = 16\left(1 - \frac{x^2}{25}\right)$
$(\text{DIST})^2 = (x-3)^2 + y^2 = (x-3)^2 + 16 - \frac{16}{25}x^2 = F(x)$
INTERVAL IS $-5 \leq x \leq 5$.

CRIT PTS? $F'(x) = 2(x-3) - \frac{32}{25}x = 0$

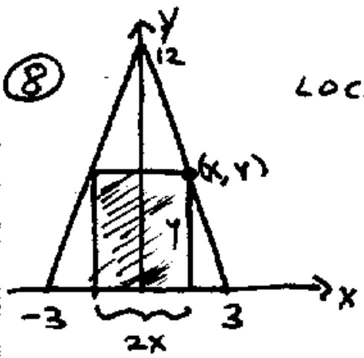
$$\frac{18}{25}x - 6 = 0$$

$$x = \frac{150}{18} = 8\frac{1}{3}.$$

OUT OF INTERVAL
SO $(\text{DIST})^2$ MINIMIZED AT END POINT.

CLEARLY $x = 5$ and POINT $(5,0)$ on ellipse will give a minimal DISTANCE OF 2 units.

(8)



LOCATE ISOSCELES TRIANGLE WITH y-axis bisecting it and base along x-axis.

$A = 2xy$, but (x,y) on Line $\frac{x}{3} + \frac{y}{12} = 1$

$\Rightarrow y = 12\left(1 - \frac{x}{3}\right)$

$y = 12 - 4x$

$\Rightarrow A(x) = 2x(12 - 4x)$

$= 24x - 8x^2$

$0 \le x \le 3$

$A'(x) = 24 - 16x = 0 \Rightarrow x = \frac{3}{2}$ inches

This gives $y = 6$ inches and a MAXIMAL AREA OF 18 in^2.

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(9) Let x = no. of doctors

y = no. of nurses

$10000x + 4000y = 100000$ is budget constraint.

$\Rightarrow 5x + 2y = 50 \qquad y = \frac{50 - 5x}{2}$

N = no. of patients to be seen.

$N = Kxy = Kx\left(\frac{50 - 5x}{2}\right) = \frac{K}{2}(50x - 5x^2)$ (K a proportionality constant)

$= N(x)$

BOUNDS ARE $x = 0$ DOCTORS OR no nurses $\Rightarrow 5x = 50 \Rightarrow x = 10$ DOCTORS

CRITICAL PTS: $N'(x) = \frac{K}{2}(50 - 10x) = 0 \Rightarrow$ $\boxed{\begin{array}{l} x = 5 \text{ DOCTORS} \\ y = 12.5 \text{ NURSES} \end{array}}$

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(10) n = # of articles produced $\quad 0 \le n \le 25$

Profit $= P =$ Revenue $-$ COST $= n(110 - 2n) - (600 + 10n + n^2)$

So $P(n) = 110n - 2n^2 - 600 - 10n - n^2 = -3n^2 + 100n - 600$

$P'(n) = -6n + 100 = 0 \Rightarrow n = \frac{100}{6} = 16\frac{2}{3}$.

Realistically, we must decide if $n = 16$ or $n = 17$ gives most profit.

$P(16) = \$232 \quad P(17) = \$233$, SO MAKE 17 ARTICLES.

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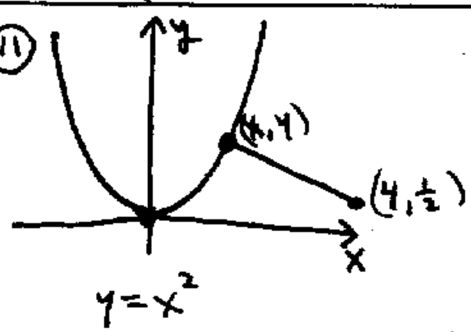
(11)



MINIMIZE $(\text{DISTANCE})^2$

$(\text{DIST})^2 = (x-4)^2 + \left(y - \frac{1}{2}\right)^2 = (x-4)^2 + \left(x^2 - \frac{1}{2}\right)^2 = F(x)$

$F'(x) = 2(x-4) + 2\left(x^2 - \frac{1}{2}\right) \cdot 2x = 0$

$2x - 8 + 4x^3 - 2x = 0 \qquad 4x^3 - 8 = 0$

$\Rightarrow x^3 = 2 \qquad x = \sqrt[3]{2}$

$\Rightarrow \left(\sqrt[3]{2}, (\sqrt[3]{2})^2\right)$

(12) Let x = number of 5% price reductions from FF 100000.

price per person $= 100000 - (.05x)100000 = 100000(1 - .05x)$

number who sign up $= 60 + 10x \le 120$ So $10x \le 60$

$0 \le x \le 6$.

$$\text{Revenue} = (60 + 10x)(100000)(1 - .05x)$$
$$= 100000(60 + 7x - .5x^2) = R(x)$$

$$R'(x) = 100000(7 - x) = 0 \Rightarrow \boxed{x = 7} \text{ no good.}$$

$\boxed{\begin{array}{c}\text{30\% Reduction} \\ \text{FF 70,000}\end{array}}$

COMPARE Revenues: $R(0) = 6,000,000$

$\boxed{R(6) = 8,400,000}$ ← THIS IS MAXIMUM FEASIBLE, SINCE PLANE CAN ONLY TAKE 120.

$R(7) = 8,450,000$

(13) 

$$\text{Area} = (2r)(h) + \tfrac{1}{2}\pi r^2$$

$$\text{Perimeter} = 2h + 2r + \pi r = 24$$

So $h = \dfrac{24 - 2r - \pi r}{2}$

Bounds: $r = 0 \to A = 0$ or $h = 0 \Rightarrow r = \dfrac{24}{2+\pi} \to A = 34.2$

$$A(r) = 2r\left(\frac{24 - 2r - \pi r}{2}\right) + \tfrac{1}{2}\pi r^2$$

$$A(r) = 24r - 2r^2 - \pi r^2 + \tfrac{1}{2}\pi r^2 = 24r - \left(\frac{\pi + 4}{2}\right)r^2$$

CRIT PT: $A'(r) = 24 - (\pi + 4)r = 0 \Rightarrow \boxed{r = \dfrac{24}{\pi + 4}} \to A = \underline{40.3}$

((14) IS BELOW)

(15) $v(t) = t^2$

But $x'(t) = v(t)$, so by the FUNDAMENTAL THEOREM OF CALCULUS,

$$x(2) - x(1) = \int_1^2 x'(t)\,dt = \int_1^2 v(t)\,dt = \int_1^2 t^2\,dt = \left.\frac{t^3}{3}\right|_1^2 = \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3}}$$

Alternatively, assume $x = 0$ when $t = 1$.

$\frac{dx}{dt} = t^2 \Rightarrow x(t) = \frac{1}{3}t^3 + C \qquad x(1) = \frac{1}{3} + C = 0 \Rightarrow C = -\frac{1}{3}$.

So $x(t) = \frac{1}{3}t^3 - \frac{1}{3}$. Hence $x(2) = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$.

⇒ NET DISPLACEMENT IS $7/3$.

(14) 

r = radius

h = height

$$\text{Area} = 2\pi r h + 2\pi r^2 = 2\pi r\left(\frac{40}{\pi r^2}\right) + 2\pi r^2 = \frac{80}{r} + 2\pi r^2 = A(r)$$

VOLUME $= \pi r^2 h = 40$

Area minimized when $A'(r) = 0$

$$h = \frac{40}{\pi r^2} \qquad A'(r) = -\frac{80}{r^2} + 4\pi r = 0$$

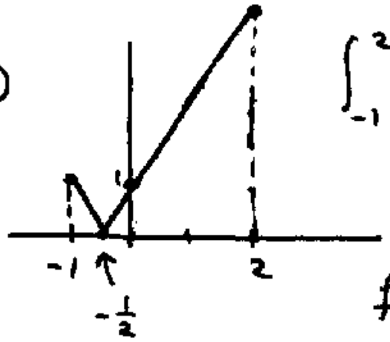
$$\Rightarrow 4\pi r = \frac{80}{r^2}$$

$$r^3 = \frac{80}{4\pi} = \frac{20}{\pi}$$

$\boxed{r = \sqrt[3]{20/\pi}} \cong \underline{1.85}$

$\boxed{h = \dfrac{40}{\pi}\left(\dfrac{\pi}{20}\right)^{2/3}} \cong \underline{3.71}$

(16)



$\int_{-1}^{2} |2x+1|\,dx$ = area under graph (TWO TRIANGLES)

$= \frac{1}{2}\left(\frac{1}{2}\right)(1) + \frac{1}{2}\left(\frac{5}{2}\right)(5) = \frac{1}{4} + \frac{25}{4} = \frac{13}{2}$

$= \boxed{6.5}$

OTHERWISE, NOTE THAT

$$f(x) = \begin{cases} -2x-1 & -1 \le x \le -\frac{1}{2} \\ (2x+1) & -\frac{1}{2} \le x \le 2 \end{cases}$$

$$\therefore \int_{-1}^{2} |2x+1|\,dx = \int_{-1}^{-\frac{1}{2}} (-2x-1)\,dx + \int_{-\frac{1}{2}}^{2} (2x+1)\,dx$$

$$= \left[-x^2 - x\right]_{-1}^{-1/2} + \left[x^2 + x\right]_{-1/2}^{2} = \left[\left(-\frac{1}{4} + \frac{1}{2}\right) - (-1+1)\right] + \left[(4+2) - \left(\frac{1}{4} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{4} + \frac{25}{4} = \frac{13}{2} = \boxed{6.5}$$

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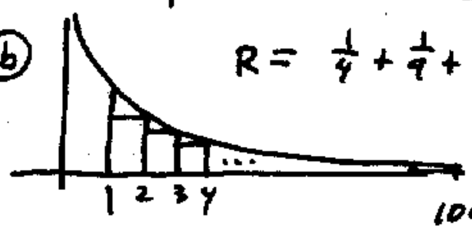
(17) (a) $I = \int_{1}^{100} \frac{dx}{x^2} = -\frac{1}{x}\Big]_{x=1}^{x=100} = -\frac{1}{100} + 1 = \boxed{\frac{99}{100}}$ = AREA UNDER GRAPH OF $f(x) = \frac{1}{x^2}$

(b)



$R = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\cdots + \frac{1}{100^2}$

(c) SINCE graph is decreasing, Right-hand sums will give an underestimate for area, so $R < I$.

(d) $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{99^2} + \frac{1}{100^2}$

$= 1 + \left(\frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{99^2} + \frac{1}{100^2}\right)$

$= 1 + R < 1 + I = 1 + \frac{99}{100} = 1.99 < 2$

So this sum is less than 2.