MATH 1a EXAM 1 REVIEW PROBLEMS — SOLUTIONS.

1 (a) $g(t) = \tan t \ \ln(2+e^{3t})$

$g'(t) = \tan t \cdot \left(\frac{3e^{3t}}{2+e^{3t}}\right) + \sec^2 t \cdot \ln(2+e^{3t})$

(b) $f(x) = \tan^{-1}\left(\frac{x-1}{x+1}\right)$ $\left(\text{Recall that } \frac{d}{dx}\left[\tan^{-1} x\right] = \frac{1}{1+x^2}\right)$

$f'(x) = \frac{1}{1+\left(\frac{x-1}{x+1}\right)^2} \cdot \left[\frac{(x+1)\cdot 1 - (x-1)\cdot 1}{(x+1)^2}\right] = \frac{2}{(x+1)^2 + (x-1)^2}$

$= \frac{2}{x^2+2x+1+x^2-2x+1} = \frac{2}{x^2+2}$

(c) $f(x) = \cos^2\left(\frac{3}{x}\right) = \left[\cos\left(\frac{3}{x}\right)\right]^2$

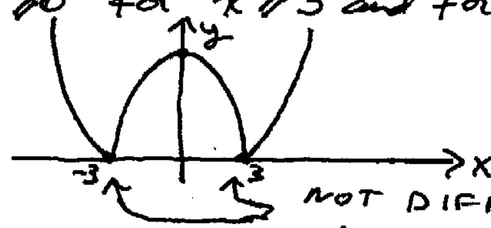
$f'(x) = 2\left[\cos\left(\frac{3}{x}\right)\right]\left[-\sin\left(\frac{3}{x}\right)\right]\cdot\left(\frac{-3}{x^2}\right) = \frac{-6\sin\left(\frac{3}{x}\right)\cos\left(\frac{3}{x}\right)}{x^2}$

(d) $k(x) = |x^2-9|$

$x^2-9<0$ for $x \in (-3,+3)$

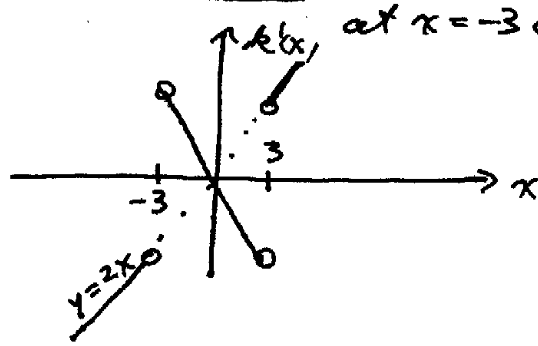
$x^2-9 \geq 0$ for $x \geq 3$ and for $x \leq -3$.

$$= \begin{cases} x^2-9 & x \leq -3 \\ -(x^2-9) & -3 < x < +3 \\ x^2-9 & x \geq 3 \end{cases}$$



NOT DIFFERENTIABLE at $x=-3$ or at $x=+3$

$$k'(x) = \begin{cases} 2x & x<-3 \\ -2x & -3<x<+3 \\ 2x & x>+3 \end{cases}$$



(e) $f(x) = 3x^{-\frac{1}{3}} - x - \frac{1}{x}$

$f'(x) = -x^{-4/3} - 1 + \frac{1}{x^2}$

$= -\frac{1}{x^{4/3}} - 1 + \frac{1}{x^2}$

$f'(1) = -1 - 1 + 1 = -1$

(f) $g(x) = \frac{x^2}{\cos(\cos x)}$

$g'(x) = \frac{\cos(\cos x)\cdot 2x - x^2(-\sin(\cos x)\cdot(-\sin x))}{[\cos(\cos x)]^2}$

$= \frac{2x\cos(\cos x) - x^2 \sin x \sin(\cos x)}{[\cos(\cos x)]^2}$

So $g'(\pi/2) = \frac{\pi\cdot 1 - \frac{\pi^2}{4}\cdot 1 \cdot 0}{1} = \boxed{\pi}$

(2) In a problem like this, there are several possible ways to make these estimates. For example, using midpoints we might deduce that $f'(x)$ is approx:

x	0.3	0.5	0.7	0.9
$f'(x)$	25	10	-30	-30

$\frac{5}{.2} = 25$ $\quad \frac{2}{.2} = 10$

$\frac{-6}{.2} = -30$ $\quad \frac{-6}{.2} = -30$

So $f'(6)$ might be in vicinity of avg. of 10 and $-30 \Rightarrow \boxed{\approx 10}$.

For $f''(6)$, estimate by $\frac{f'(.7) - f'(.5)}{.2} = \frac{-30-10}{.2} = \frac{-40}{.2} = \boxed{-200}$

These are very rough estimates, of course.

(3) (a) For $x \neq 0$, we have $f'(x) = x^2 \cdot \cos\left(\frac{1}{x}\right)\cdot\left(\frac{-1}{x^2}\right) + 2x \sin\left(\frac{1}{x}\right)$

$= -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right)$

(b) $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \left(\frac{x^2 \sin\left(\frac{1}{x}\right)}{x}\right) = \lim_{x \to 0} \left[x \cdot \sin\left(\frac{1}{x}\right)\right]$.

2nd factor bounces around between -1 and $+1$, but first factor tends toward 0. $\Rightarrow \boxed{\text{LIMIT} = 0}$

(MORE PRECISELY, $-1 \leq \sin\left(\frac{1}{x}\right) \leq +1$

$-|x| \leq x \sin\left(\frac{1}{x}\right) \leq +|x|$, with a little thought.

Since $\lim_{x \to 0}(-|x|) = 0 = \lim_{x \to 0}(+|x|)$, SQUEEZE THEOREM IMPLIES THAT $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$ also.)

(4) Bob: $\frac{fg' + gf' - 2fg'}{g^2} = \frac{gf' - fg'}{g^2}$ RIGHT!

CAROL: $f' \cdot \left(\frac{g \cdot 0 - 1 \cdot g'}{g^2}\right) = \frac{-f'g'}{g^2}$ WRONG!

TED: WRONG! ALICE: $\frac{f' - \frac{fg'}{g}}{g} \cdot \frac{g}{g} = \frac{gf' - fg'}{g^2}$ RIGHT!

(5) (a) YES. (2h instead of h) (c) YES (STILL Finding slope of Tangent line, just using pts to left + right of a)

(b) NO (d) YES (e) YES.

(6) $f'(0) = \lim_{x \to 0}\left(\frac{f(x) - f(0)}{x - 0}\right) = \lim_{x \to 0}\left(\frac{\frac{1}{2x+1} - 1}{x}\right) = \lim_{x \to 0}\left(\frac{1 - (2x+1)}{x(2x+1)}\right)$

$= \lim_{x \to 0}\left(\frac{-2x}{x(2x+1)}\right) = \lim_{x \to 0}\left(\frac{-2}{2x+1}\right) = \frac{-2}{1} = \boxed{-2}$

A GOOD PROTOCOL FOR RELATED RATES PROBLEMS (LIKE THIS) is :

#1 – DRAW PICTURE

#2 – LABEL Relevant variables

#3 – Relate VARIABLES (using things like Pythag. thm, SIMILAR TRIANGLES, Geometric formulas, etcetera.

#4 – Take derivative with respect to t. $\Rightarrow$ Relates Rates.

#5 – PLUG IN NUMBERS AND solve for RATE.

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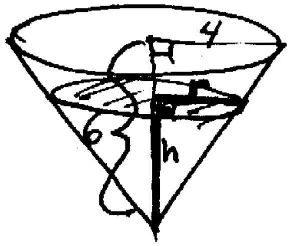

By similar triangles, $\frac{r}{h} = \frac{4}{6} \Rightarrow \boxed{r = \frac{2}{3}h}$

Geometric formula $V = \frac{1}{3}\pi r^2 h$

$$\Rightarrow V = \frac{1}{3}\pi\left(\frac{4}{9}h^2\right)h = \frac{4}{27}\pi h^3$$

we're given that $\frac{dV}{dt} = -2$ in³/min, want to find $\frac{dh}{dt}$.

Differentiate w.r.t t $\Rightarrow \boxed{\frac{dV}{dt} = \frac{4}{9}\pi h^2 \frac{dh}{dt}}$

we're interested in what's happening when $h = 3$ in.

$$\Rightarrow -2 = \frac{4}{9}\pi \cdot 9 \cdot \frac{dh}{dt} \Rightarrow -2 = 4\pi \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = -\frac{1}{2\pi} \text{ in/min.}$$

So height is dropping at about 0.16 in/min at that point.

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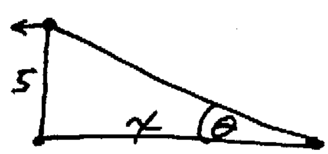


AIRPORT

x = DISTANCE FROM AIRPORT

$\frac{dx}{dt} = 507$ mi/hr Find $\frac{d\theta}{dt}$

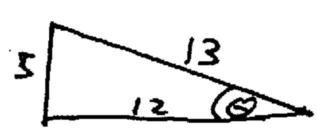
(WORDING A LITTLE VAGUE ON WHAT ANGLE IS, BUT ABOVE θ seems right)

using tangent, we get

$$\tan\theta = \frac{5}{x}$$

$$\text{So } \sec^2\theta \cdot \frac{d\theta}{dt} = -\frac{5}{x^2} \cdot \frac{dx}{dt}$$

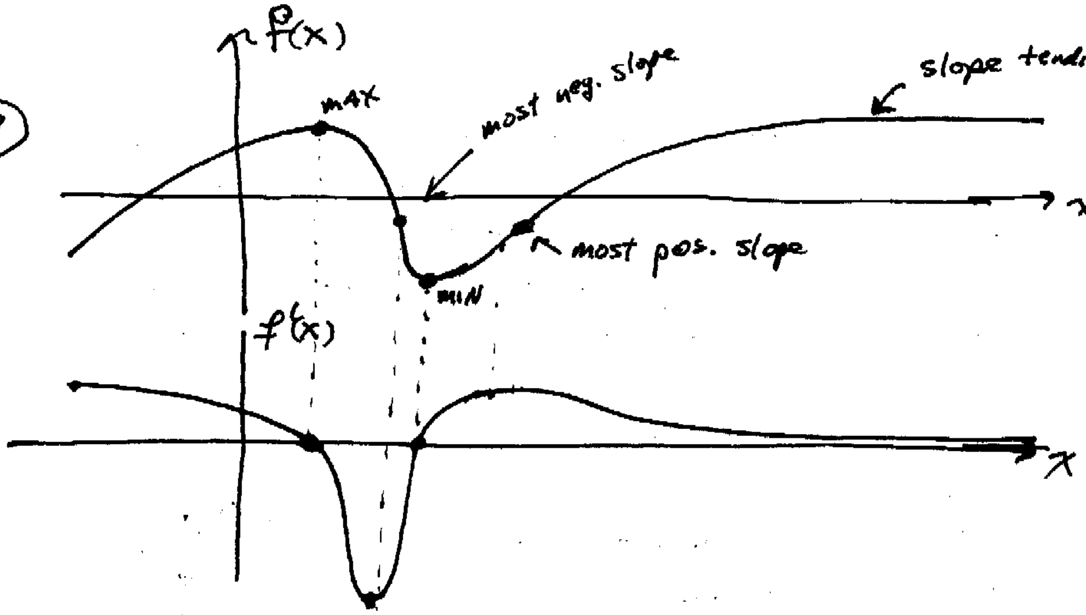
At point in question, $x = 12$ mi $\Rightarrow$



So $\sec\theta = \frac{13}{12}$

$$\Rightarrow \frac{169}{144} \cdot \frac{d\theta}{dt} = \frac{-5}{144}(507) \Rightarrow \frac{d\theta}{dt} = \frac{-5(507)}{169} = -15 \text{ radians/sec}$$

(9)



(10) $s(t) = \frac{2}{3}t^3 - 7t^2 + 20t + 8$

(a) $v(t) = 2t^2 - 14t + 20$

(b) $a(t) = 4t - 14$

(c) moving Right when $v > 0$

$\Rightarrow 2(t^2 - 7t + 10) > 0$

$(t-2)(t-5) > 0$

$$\begin{array}{cccccc} & + & & - & & + \\ 0 & & 2 & & 5 & \end{array} \quad v$$

$\Rightarrow \underline{0 < t < 2}$ or $\underline{t > 5}$.

(d) velocity decreasing when

$v'(t) < 0 \Rightarrow$ when $a(t) = 4t - 14 < 0 \Rightarrow t < \frac{7}{2}$

$\Rightarrow \boxed{t < 3.5}$

(e) OBJECT CHANGES DIRECTION WHEN VELOCITY CHANGES FROM + to − $\Rightarrow$ $\boxed{t = 2 \text{ and } t = 5}$

(11) $x^2y + 2y^3 = 3x + 2y + 54$ POINT $(x, y) = (2, 3)$

Differentiate implicitly:

$x^2 \cdot \frac{dy}{dx} + 2xy + 6y^2 \cdot \frac{dy}{dx} = 3 + 2\frac{dy}{dx}$

$\Rightarrow (x^2 + 6y^2 - 2)\frac{dy}{dx} = 3 - 2xy$

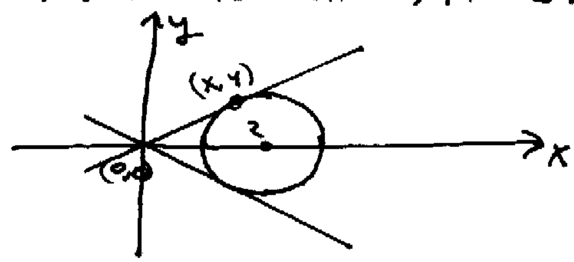
$$\frac{dy}{dx} = \frac{3 - 2xy}{x^2 + 6y^2 - 2} \qquad \left.\frac{dy}{dx}\right|_{(2,3)} = \frac{3 - 12}{4 + 54 - 2} = \frac{-9}{56} = m$$

With point $(2,3)$ $\Rightarrow$ Tangent line given by

$$\boxed{y - 3 = -\frac{9}{56}(x - 2)}$$

(12) though you don't need to know that this is a circle, it is.

$$x^2-4x \qquad +y^2=-3$$
$$x^2-4x+4+y^2=-3+4$$
$$(x-2)^2+y^2=1$$



From $x^2-4x+y^2+3=0$, we get $2x-4+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{4-2x}{2y}$

slope of TL $=\frac{y-0}{x-0}=\frac{y}{x}$

So $\frac{y}{x}=\frac{4-2x}{2y} \Rightarrow 2y^2=4x-2x^2$
$\Rightarrow x^2-2x+y^2=0$

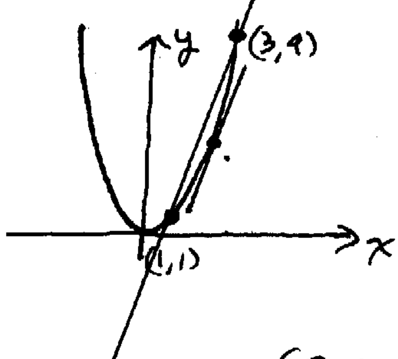
$$\left\{\begin{array}{l} x^2-4x+y^2+3=0 \\ x^2-2x+y^2\quad\;\;=0 \end{array}\right\} \Rightarrow 2x-3=0 \text{ (by subtracting equations).}$$
$$\Rightarrow x=\frac{3}{2}$$

So $\frac{9}{4}-6+y^2+3=0 \Rightarrow y^2=\frac{3}{4} \Rightarrow y=\pm\frac{\sqrt{3}}{2}$.

So slope $=\frac{\frac{\sqrt{3}}{2}}{\frac{3}{2}}=\frac{\sqrt{3}}{3}$ or $\frac{-\frac{\sqrt{3}}{2}}{\frac{3}{2}}=-\frac{\sqrt{3}}{3}$.

$\Rightarrow$ Lines are $\boxed{y=\frac{\sqrt{3}}{3}x}$ or $\boxed{y=-\frac{\sqrt{3}}{3}x}$

(13)



SLOPE FROM (1,1) to (3,9)

is $\frac{9-1}{3-1}=\frac{8}{2}=4$

$y=x^2 \Rightarrow \frac{dy}{dx}=2x$

so we want $2x=4 \Rightarrow x=2$

$\Rightarrow$ $\boxed{\text{POINT is } (2,4)}$

(14) To Approximate $\sqrt[5]{31}$, note first that $\sqrt[5]{32} = 2$.
Use LINEAR APPROXIMATION of $f(x) = \sqrt[5]{x}$ with $a = 32$.

$$f(x) \cong f(32) + f'(32)(x-32)$$

$f(x) = \sqrt[5]{x} = x^{1/5}$ $\qquad$ $f(32) = 2$

$f'(x) = \frac{1}{5} x^{-4/5} = \frac{1}{5(\sqrt[5]{x})^4}$ $\qquad$ $f'(32) = \frac{1}{5 \cdot 16} = \frac{1}{80}$

So $\sqrt[5]{x} \cong 2 + \frac{1}{80}(x-32)$ for x near 32

In particular, $\sqrt[5]{31} \cong 2 + \frac{1}{80}(31-32) = 2 - \frac{1}{80} = \frac{159}{80}$

$= 1.9875$

(Actual $\sqrt[5]{31} = 1.98734\cdots\cdots$)

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(15) If $f(x) = \ln x$ $\qquad$ $f(1) = \ln(1) = 0$

$f'(x) = \frac{1}{x}$ $\qquad$ $f'(1) = 1$

So $\ln x \cong 0 + 1(x-1) = x-1$ for x near 1.

Therefore, for x near 1 we have

$$\frac{\ln x}{x^2-1} \cong \frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}.$$

$$\text{So } \lim_{x\to 1}\left(\frac{\ln x}{x^2-1}\right) = \lim_{x\to 1}\left(\frac{1}{x+1}\right) = \frac{1}{2}$$