Math 1a Final Review Problems -- Fall 1998

The problems in this review have been taken from recent final exams and reviews from rpevious semesters.

Problems from the Spring 97 Final Exam

1) Find derivatives for the following functions (simplify completely):

a) $f(x) = 2^{(2+6x^3)}$ b) $f(\theta) = 3\cos^2(5\theta)$

c) $f(x) = \sin^{-1}(3x+1)$ d) $f(x) = \ln(1+\ln x)$

e) $f(x) = (\ln x)^{\tan^{-1} x}$ f) $f(x) = \int_2^{e^x} \frac{\ln z}{z}\, dz$

2) Find the following integrals:

a) $\int (3x + x^e + e^x)dx$ b) $\int \frac{e^x}{1+e^{2x}} dx$

c) $\int_1^e \frac{(\ln x)^2}{x} dx$ d) $\int \frac{1}{x^2+2x+3} dx$

3) Sketch the graph of $f(x) = \int_0^x (t^2 - 3t + 2)dt$, showing any inflection points, relative maxima and minima, intercepts, and asymptotes.

4) Consider the piecewise defined function

$$f(x) = \begin{Bmatrix} x^2 e^{-x} & x \geq 0 \\ -x & x < 0 \end{Bmatrix}.$$

a) Find the y-intercept of f .

b) Find the x-intercept(s) of f .

c) Find any horizontal or vertical asymptotes for the graph of f .

d) Find the coordinates of any critical point(s) of f , and determine any relative maxima or minima.

e) Find the x coordinates of any inflection point(s) of f.

f) Sketch a graph of f.

g) Is this a continuous function? Explain briefly.

h) Is this a differentiable function? Explain briefly.

5) An apple grower has an orchard with 30 trees which yields 400 apples per tree in a typical season. This grower plans to add more trees to this orchard and anticipates that for each additional tree that she plants, the yield for each tree in the orchard would be reduced by 7 apples. How many trees would produce the largest crop for this orchard? (whole trees only)

6) The police observe that the skid marks made by a stopping car are 250 feet long. Assuming the car decelerated at a constant rate of 20 ft/sec^2, skidding all the way, how fast was the car travelling when the brakes were initially applied?

7) Find the equation of the tangent line to the curve

$$\cos(xy) + e^{2y} = e^4 \text{ at the point } (\tfrac{\pi}{4}, 2).$$

8) Consider the graph of the function $f(x) = \tan^{-1} x$. Consider the region bounded below by the x-axis and above by the graph between $x = 0$ and $x = 1$. If we were to partition this region into n strips of equal length, let L(n) be the Riemann Sum approximation for the area using left-hand endpoints, R(n) be the Riemann Sum approximation for the area using the right-hand endpoints, and let T(n) be the approximation for the area of this region using n trapezoids.

Circle whether each of the statements below is TRUE or FALSE and briefly explain why.

a) L(50) < $\int_0^1 \tan^{-1} x\, dx$ TRUE or FALSE

b) R(50) ≤ T(50) TRUE or FALSE

c) $\int_0^1 \tan^{-1} x\, dx$ ≤ T(50) TRUE or FALSE

d) L(50) < L(100) TRUE or FALSE

e) π/4 < $\int_0^1 \tan^{-1} x\, dx$ TRUE or FALSE

9) A population of 200 toads is released in a remote area in Australia with an essentially unlimited food supply.

a) If the population grows exponentially according to $P(t) = Ae^{kt}$ and if 2 years after release we find that there are 800 toads, give an expression for the number of toads expected t years after they were released. In particular, determine how many toads there will be 5 years after they were released.

b) Suppose that the toad population were to spread out over time in a circular pattern so that the population density (in toads per square mile) remains constant. If the radius of the colony was initially 1 mile, how large will the radius of this circular colony be five years after release?

c) At what rate will the radius of the colony be expanding five years after release?

Other problems from past finals and reviews

10) In each of the following, find $f'(x)$ or $\frac{dy}{dx}$:

a) $f(x) = e^{\tan^{-1}(\ln x)}$ b) $x^2y^3 + 3xe^y - 4\ln x = 5$

c) $y = \int_0^{\ln(4p)} \sin(e^x)dx$ d) $f(x) = \left[x^2 e^{3x} + \ln x\right]^5$

e) $f(x) = x^{\ln x}$ f) $f(x) = 7^{-4x}$

g) $f(x) = \frac{e^{2x}-1}{e^{2x}+1}$ at $x = 0$.

11) Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.

12) Evaluate the twenty-first (21st) derivative of

a) $p = 3\cos x$ b) $u = 3e^{2x}$

c) $q = 3x^2 + 2x^5 - \frac{x^{12}}{17}$

13) A small tired monkey is climbing up the y axis, with acceleration $a(t) = -\frac{1}{(t+1)^2}$ meters/sec^2.

At time $t = 0$ the monkey finds itself at $y = 1$, moving up with a speed of 1 meter/sec. When will the monkey reach the point $y = 30$? Does your answer make sense from a practical point of view?

14) A rope is fastened to the ground at a distance of eight feet from a vertical wall. A monkey weighing 37 pounds climbs the wall holding onto the loose end of the rope, holding it taut. At which rate is the length of the rope (between the monkey's hand and where it is fastened to the ground) increasing when the monkey is six feet up the wall, climbing at a rate of four feet per second?

15) Evaluate the following integrals:

a) $\int_{e^2}^{e^3} (\frac{3}{x} + \frac{2x}{e^6 - e^4})dx$ b) $\int \frac{2x+3}{\sqrt{4-x^2}}dx$

c) $\int_0^{\sqrt{p}} x\sin(x^2)dx$ d) $\int \frac{1}{e^{3t}}dt$

e) $\int_1^2 \frac{x^3+1}{x^2}dx$ f) $\int_1^2 \frac{\ln x}{x}dx$

g) $\int \sin^2(3x)\cos(3x)dx$ h) $\int \frac{dx}{4+9x^2}$

16) Two commercial jets at 40,000 ft are flying at 520 mph along straight line courses that cross at right angles. How fast is the distance between the planes closing when plane A is 5 mi from the intersection point and plane B is 12 miles from the intersection point?

17) If $\int_0^x f(t)dt = 3x^2 + e^x - \cos x$, then $f(2)$ =_ ?

18) Use linear approximation to estimate cos 31.8° to three decimal places. If needed, you may use the approximations $\pi \approx 3.1416$ and $\sqrt{3} \approx 1.732$.

19) A right triangle whose hypotenuse is $\sqrt{3}$ m long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made in this way. (It may be helpful to know that the volume of a right circular cone is one-third the area of its base times its height.)

20) Let $f(x) = e^{2x} - 8e^x + 12$

a) Find the y-intercept of f.
b) Find the x-intercept(s) of f
c) Find the coordinates of any critical point(s) of f
d) Find the coordinates of any inflection point(s) of f.
e) State the behavior of f as $x\to\infty$ and as $x\to-\infty$
f) Sketch a graph of f, labeling the above features.

21) Given the function $f(x) = x^4 - \frac{x^3}{3}$, graph not only this function but also the graphs of its first and second derivative functions.

22) Sketch the graph of $y = \frac{x^2 - x + 1}{x - 1}$, showing any inflection points, relative maxima and minima, intercepts, and asymptotes. Indicate with arrows those intervals where the graph is concave up and concave down.

23) Graph the function $f(x) = \frac{x-1}{x^2}$ showing intercepts, critical points, relative maxima and minima, asymptotes, points of inflection, etcetera. That is, find all this information and show it on the graph.

24) A function $f(x)$ and its first and second derivatives are given below. Answer a) through e).

$$f(x) = x\sqrt[3]{x-4} \qquad f'(x) = \frac{4x-12}{3(x-4)^{2/3}}$$

$$f''(x) = \frac{4x-24}{9(x-4)^{5/3}}$$

a) Identify any x or y intercepts for the graph of this function.

b) Find any critical points and determine whether they are relative maxima or minima or neither.

c) Find any points of inflection.

d) Find any vertical or horizontal asymptotes for this graph.

e) Give a good sketch of the graph of this function, illustrating clearly all of the above features.

25) A company wants to make a closed box (6 sides) whose volume is $\frac{10}{3}$ cubic feet. The base of the box will be a rectangle whose length is twice its width. The material for the sides and the top of the box costs 20 cents per square foot. The bottom requires heavier material which costs 30 cents per square foot. The company wishes to make the box as cheaply as possible. What are the dimensions of the cheapest such box, and what is its cost?

26) Graph the function $y = 3x^{5/3} - 5x$. Indicate inflection points, local maxima and minima, and intercepts.

27) A starting airplane has a constant acceleration while moving down the runway, from rest. The plane takes off with a speed of 100 miles/hr, after running 1 mile. What is the acceleration, and how long does the plane move on the runway?

28) Water is being pumped from an inverted right circular conical tank at the rate of 36 ft^3/min. The tank, which stands vertex down and base up, has a height of 12 ft and a base diameter of 16 ft. How fast is the water level dropping when the water is 9 ft deep?

29) A train and a balloon leave a given point at the same moment. The train moves in a straight line at a steady speed of 50 km/hr. The ballon rises vertically at a steady rate of 10 km/hr. At what speed are they moving apart?

30) A car is travelling at 200 feet/sec on a road with a speed limit of 100 feet/sec (approx 68 miles/hour) when the driver learns of a police car hiding up ahead. The driver applies the brakes in such a way that the car undergoes a constant deceleration of 10 feet/sec^2. If the police car is 1280 feet ahead, will the driver be within the speed limit when passing the police car? What will be the velocity of the car as it passes the police car?

31) Evaluate the following limits:

a) $\lim_{x\to\infty} 2x\ln(1+\frac{3}{x})$ b) $\lim_{x\to 0}\left(\frac{xe^x - x}{1-\cos x}\right)$

c) $\lim_{x\to 0}(1+\sin 2x)^{1/x}$ d) $\lim_{x\to 0}\frac{\sec x - 1}{x^2}$

e) $\lim_{x\to 0}\frac{x\sin x}{\sin(x^2)}$ f) $\lim_{x\to 0}\frac{x(e^x-1)}{e^x - x - 1}$

32) a) Find $\lim_{x\to\infty}(x^{2/x})$ b) Find $\frac{d}{dx}(x^{2/x})$

33) a) Find $\int_{-2}^{2}|x-1|\,dx$

b) Find the average value of the function $f(x) = |x-2|$ on the interval $[0,5]$.

34) A car is travelling at 25 meters/second when the driver applies the brakes to avoid hitting a child. During this braking maneuver the car experiences a constant deceleration of 5 meters/sec^2. How long does it it take the car to come to a stop, and how far does it travel before stopping?

35) Find the dimensions of the rectangle of largest area, with sides parallel to the coordinate axes, that can be inscribed inside the ellipse $\frac{x^2}{4} + y^2 = 1$.

36) If $f(x) = \begin{cases} x+3 & \text{for } x<0 \\ x^2 & \text{for } x \ge 0 \end{cases}$, find $\int_{-3}^{2} f(x)dx$.

37) For the integral $I = \int_0^2 \sqrt{1+x^2}\,dx$, using n = 2 subdivisions, let *L* denote the Riemann Sum using left-hand endpoints of each interval, *R* the Riemann Sum using right-hand endpoints of each interval, *M* the Riemann Sum using the midpoint of each interval, *T* the result using the Trapezoid Rule, and *S* the result using Simpson’s Rule (with n = 4).

a) Using the values $f(0)=1$, $f(.5)=1.118$, $f(1)=1.414$, $f(1.5)=1.803$, and $f(2)=2.236$, compute each of the quantities *L*, *R*, *M*, and *T* using n = 2 subdivisions, and *S* using n = 4 subdivisions. (Left endpoints, Right endpoints, Midpoints, Trapezoids, and Simpson's Rule)

b) List *L*, *R*, *M*, *T* and *S* in increasing order:
_____ < _____ < _____ < _____ < _____

c) By examining the graph of $f(x) = \sqrt{1+x^2}$ on the interval [0,2] or the derivatives

$$f'(x) = \frac{x}{\sqrt{x^2+1}} \text{ and } f''(x) = \frac{1}{(x^2+1)^{3/2}},$$

determine whether *L*, *R*, and *T* are underestimates or overestimates for the exact value *I* of this definite integral. Briefly explain each conclusion.

38) A car is driving south on US Route 95. It moves so that at time *t* , its acceleration is (30 sin *t*) ft/sec^2, where *t* is in seconds. At time *t* = 0, its velocity is 30 ft/sec.

a) Find the car’s velocity as a function of time.
b) How far did the car travel between t=0 and t=3π?
c) At *t*=10π , a policeman, who has been observing the car for some time, stops the car and arrests the driver for reckless driving. Explain in ordinary English why this is appropriate.
d) Should the charge also include speeding? (the speed limit is 55 mi/hr or 80.7 ft/sec.)

39) An architect wants to design a window in the shape of a rectangle capped by a semicircle. If the perimeter of the window is constrained to be 24 feet, what dimensions should the architect choose for the window in order to admit the greatest amount of light?

40) A ladder 13 feet long is leaning against the side of a building. If the foot of the ladder is pulled away from the building at the rate of 0.1 foot per second, how fast is the angle formed by the ladder and the ground changing at the instant when the top of the ladder is 12 feet above the ground?

41) The displacement of an object moving along a line is given by the function $s(t) = -2t^3 - 3t^2 + 12t + 40$. Compute the <u>total distance traveled</u> from $t = 0$ to $t = 3$.

42) Consider the function $f(x) = x^3 e^{-x}$.

a) Over what interval(s) is this function **decreasing**?

b) At which value of x is this function **decreasing most rapidly**?

43) A highway patrol plane flies 3 mi above a level, straight road at a steady ground speed of 120 mph. The pilot sees an oncoming car and determines with radar that the line-of sight distance from the plane to the car is 5 mi and decreasing at the rate of 160 mph. Find the car's speed along the highway.

44) If f(x) is continuous on the interval [0,1], what is the relation between $\int_0^1 f(x)dx$ and $\int_0^1 f(1-x)dx$? Explain your answer.

45) A car is accelerating from a complete stop to its maximum velocity. Measurements indicate that the car's acceleration (in ft/sec^2) at any time t (in seconds) is well-approximated by the formula:

$$a(t) = \begin{cases} 15 - t & \text{for} \quad 0 \le t \le 15 \\ 0 & \text{for} \quad t > 15 \end{cases}$$

a) Sketch a graph of $y = a(t)$ and of $y = v(t)$, the velocity of the car. Label important features on each graph.

b) Give a formula for $v(t)$.

c) What is the max. velocity the car attains for $t > 0$?

46) If $f(x) = \begin{cases} x+3 & \text{for } x < 0 \\ x^2 & \text{for } x \ge 0 \end{cases}$, find $\int_{-3}^{2} f(x)dx$.

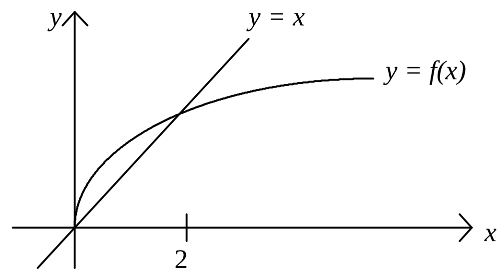
47) Find the equations for two lines through the origin that are tangent to the curve $x^2 - 4x + y^2 + 3 = 0$

48) Find the equation of the line tangent to the graph of $f(x) = 2\sin x - 3\cos x$ at $x = \frac{3p}{2}$.

49) Find the equation of the tangent line to the curve $x^2y + 2y^3 = 3x + 2y + 54$ at the point where $(x, y) = (2,3)$.

50) Based on the figure below, rank the following quantities from least to greatest indicating whether any are equal:

0 , 1 , 4 , $f(2)$, $f'(2)$, $f''(2)$, $\int_0^2 f'(x)dx$, $\int_0^2 f(x)dx$



51) You are sitting in the grandstand at an auto racetrack. Your seat is 100 feet from the track. A car races by at 150 mi/hr (220 ft/sec) and you turn your head to follow the car.

a) How fast is your head turning (in radians per second) at the point when the car is right in front of you?

b) How fast is your head turning when the car is 240 feet down the track (a little more than a second later)?

52) A tanker accident has spilled oil in Pristine Bay. Oil-eating bacteria are gobbling up 5 cubic feet of oil per hour. The oil slick has the form of a circular disk of uniform thickness. When the radius of the disk is 500 feet, the thickness of the slick is 0.01 feet and is decreasing at a rate of 0.001 feet per hour.

a) At what rate is the area of the slick changing at this time?

b) Is the area of the slick increasing or decreasing then?

53) It's 9:30 p.m. Your snowmobile is out of gas and you are three miles due south of a major highway. The nearest service station on the highway is six miles east of your position; it closes at midnight. You can walk four miles per hour on roads but only three miles per hour through snowy fields.

a) What route is best for minimizing time?

b) Can you make it to the service station before it closes?

54)a) Find a function $F(x)$ such that $F'(x)=\sin(2x)+\cos(2x)$ and such that $F(0) = 0$.

b) Find a function $F(x)$ such that $F'(x) = (1-x)\sqrt{x}$ and such that $F(0)=0$.

c) If $f'(x) = 2x - 3$ and $f(1)=0$, for what other x is $f(x)=0$?