MATH 1a FINAL REVIEW PROBLEMS – SOLUTIONS   FALL 1998

(1) (a) $f'(x) = \left(2^{2+6x^3}\right)(\ln 2)(18x^2)$

(b) $f'(\theta) = 6(\cos 5\theta)(-\sin 5\theta)(5)$
$= -30\cos 5\theta \sin 5\theta$

(c) $f'(x) = \dfrac{3}{\sqrt{1-(3x+1)^2}}$

(d) $f'(x) = \dfrac{\frac{1}{x}}{1+\ln x} = \dfrac{1}{x(1+\ln x)}$

(e) write $y = (\ln x)^{\tan^{-1}x}$ and use logarithmic differentiation

$\ln y = \tan^{-1}x \cdot \ln(\ln x)$

$\frac{1}{y}\frac{dy}{dx} = \tan^{-1}x \cdot \frac{1}{x\ln x} + \frac{1}{1+x^2}\cdot \ln(\ln x)$

$\frac{1}{y}\frac{dy}{dx} = \frac{\tan^{-1}x}{x\ln x} + \frac{\ln(\ln x)}{1+x^2}$

$\frac{dy}{dx} = f'(x) = \left[\frac{\tan^{-1}x}{x\ln x} + \frac{\ln(\ln x)}{1+x^2}\right](\ln x)^{\tan^{-1}x}$

(f) $f'(x) = \frac{d}{dx}\int_2^{e^x} \frac{\ln z}{z}\,dz$

$= \frac{\ln(e^x)}{e^x}\cdot\frac{d}{dx}(e^x) - \frac{\ln 2}{2}\cdot\frac{d}{dx}(2)$
(by Leibnitz' Rule)

$= \frac{x}{e^x}\cdot e^x - 0 = \boxed{x}$

(2) (a) $\frac{3x^2}{2} + \frac{x^{e+1}}{e+1} + e^x + C$

(b) $\int \frac{e^x\,dx}{1+(e^x)^2}$ $u = e^x$, $du = e^x dx$

$= \int \frac{du}{1+u^2} = \tan^{-1}u + C$

$= \tan^{-1}(e^x) + C$

(c) $\int_1^e \frac{(\ln x)^2}{x}dx$ $u = \ln x$, $du = \frac{dx}{x}$

$= \int_0^1 u^2\,du = \frac{u^3}{3}\Big|_0^1 = \frac{1}{3}$

(d) $\int \frac{dx}{(x+1)^2+2} = \frac{1}{2}\int \frac{dx}{\left(\frac{x+1}{\sqrt{2}}\right)^2+1}$

$u = \frac{x+1}{\sqrt{2}}$ $du = \frac{dx}{\sqrt{2}}$ $\Rightarrow \frac{1}{\sqrt{2}}\int\frac{du}{u^2+1} = \frac{1}{\sqrt{2}}\tan^{-1}u = \boxed{\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+C}$

(3) $f(x) = \int_0^x (t^2-3t+2)\,dt$

$= \frac{t^3}{3} - \frac{3t^2}{2} + 2t\Big]_{t=0}^{t=x}$

$f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x$

$f'(x) = x^2 - 3x + 2$

$f''(x) = 2x - 3$

X-Intercepts: $x\left(\frac{1}{3}x^2 - \frac{3}{2}x + 2\right) = 0$

$\frac{1}{6}x(2x^2 - 9x + 12) = 0$

$\boxed{x=0}$ $x = \frac{9 \pm \sqrt{81-96}}{4}$

$\Rightarrow (0,0)$ only.

CRIT PTS: $(x-2)(x-1) = 0$

$\boxed{x=1}$ $\boxed{x=2}$

$f''(1) = -1 \Rightarrow$ Rel MAX AT $x=1$

$f''(2) = +1 \Rightarrow$ Rel. MIN AT $x=2$

$f(1) = \frac{1}{3} - \frac{3}{2} + 2 = \frac{5}{6}$

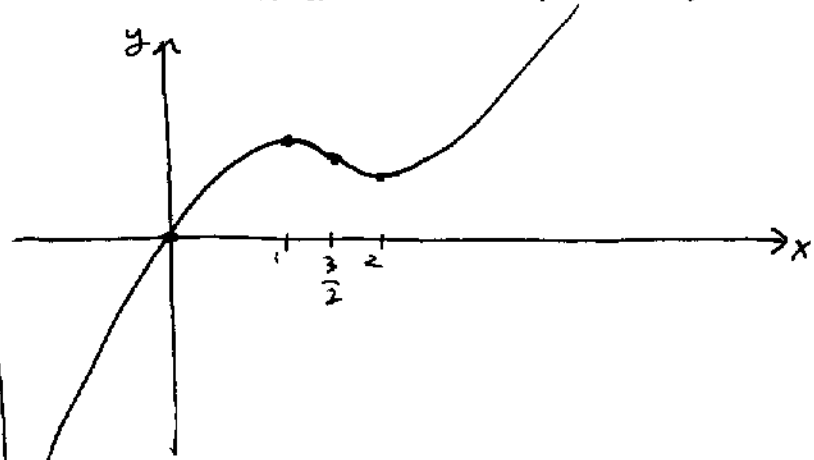
$f(2) = \frac{8}{3} - 6 + 4 = +\frac{2}{3}$

P.O.I.: $2x-3=0$ $\boxed{x=\frac{3}{2}}$

$f\left(\frac{3}{2}\right) = \frac{1}{3}\left(\frac{27}{8}\right) - \frac{27}{8} + 3$

$= \frac{72}{24} - \frac{54}{24} = \frac{18}{24} = \frac{3}{4}$

THERE ARE NO ASYMPTOTES.



④ $f(x)=x^2e^{-x} \quad x\geq 0$

$f'(x) = -x^2e^{-x}+2xe^{-x} = (2x-x^2)e^{-x}$

$f''(x) = -(2x-x^2)e^{-x}+(2-2x)e^{-x}$

$= (x^2-4x+2)e^{-x}$

$f(0)=0 \Rightarrow (0,0)$ on graph.

SINCE $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)=0$

Graph continuous at $(0,0)$.

no other x-intercepts

$\lim_{x\to+\infty}(x^2e^{-x}) = \lim_{x\to+\infty}\frac{x^2}{e^x}\to\frac{\infty}{\infty}$

$\therefore = \lim_{x\to+\infty}\left(\frac{2x}{e^x}\right) = \lim_{x\to+\infty}\left(\frac{2}{e^x}\right)=0$

So $\lim_{x\to+\infty} f(x)=0$ (Horiz Asymptote)

CRIT PTS: $2x-x^2 = x(2-x)=0$

$\Rightarrow x=0 \quad x=2$.

$f''(2) = -2e^{2}<0 \therefore$ (Rel. MAX)

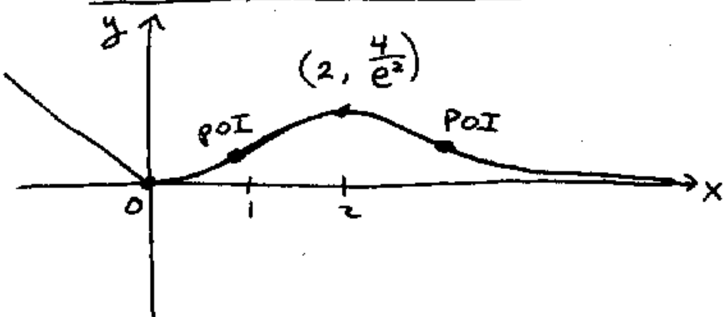
POI: $x^2-4x+2=0$

$x=\frac{4\pm\sqrt{16-8}}{2}$

$x = 2\pm\sqrt{2}$

$\lim_{x\to 0^-} f'(x) = -1 \quad \lim_{x\to 0^+} f'(x)=0$

slopes don't MATCH, so graph is not differentiable at 0



⑤ Let x = # of add'l trees to be planted.

TOTAL TREES = $30 + x$

apples per tree = $400 - 7x$

TOTAL APPLES = $f(x) = (30+x)(400-7x)$

$= 12000 + 190x - 7x^2$

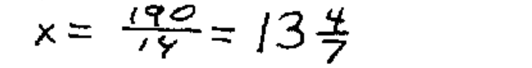
$f'(x) = 190 - 14x = 0$

$x = \frac{190}{14} = 13\frac{4}{7}$

$f(13) = (43)(309) = 13287$

$f(14) = (44)(302) = 13288$.

She should plant 14 additional trees for a total of 44 trees.

⑥ 

$a = \frac{dv}{dt} = -20 \text{ ft/sec}^2$

Let $V(0) = V_0$.

$V(t) = -20t + C$

$V(0) = C = V_0 \Rightarrow \boxed{V(t) = V_0 - 20t}$

So $\frac{dx}{dt} = V_0 - 20t$

$X(t) = V_0t - 10t^2 + C'$

we can take $X(0)=0$, and if we let T = duration of SKID we'll have $x(T) = 250$.

$x(0) = C' = 0 \Rightarrow \boxed{x(t) = V_0t - 10t^2}$

we also know that $v(T) = 0$

So $\begin{cases} V_0 - 20T = 0 \\ V_0T - 10T^2 = 250 \end{cases}$

$V_0 = 20T \Rightarrow 20T^2 - 10T^2 = 250$

$10T^2 = 250 \quad T^2 = 25$

So $T = 5$ sec and $\boxed{V_0 = 100 \text{ feet/sec}}$

⑦ $\cos(xy) + e^{2y} = e^4 \quad \left(\frac{\pi}{4}, 2\right)$

$-\sin(xy)\left[x\cdot\frac{dy}{dx} + y\right] + e^{2y}\cdot 2\frac{dy}{dx} = 0$

$\frac{dy}{dx}\left[2e^{2y} - x\sin xy\right] = y\sin(xy)$

$\frac{dy}{dx} = \frac{y\sin xy}{2e^{2y} - x\sin xy}$

$\left.\frac{dy}{dx}\right|_{(\frac{\pi}{4},2)} = \frac{2\cdot 1}{2e^4 - \frac{\pi}{4}} = \frac{8}{8e^4 - \pi}$

TANGENT LINE IS:

$\boxed{y - 2 = \left(\frac{8}{8e^4-\pi}\right)\left(x - \frac{\pi}{4}\right)}$

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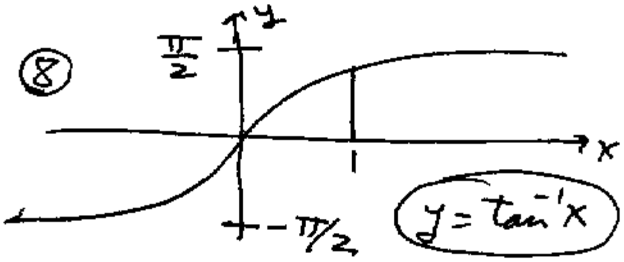

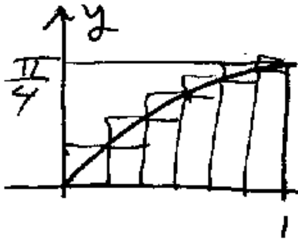

Graph is concave down, and Increasing

(a) $L(50)$ underestimates $\Rightarrow$ $\boxed{\text{TRUE}}$

(b) $R(50)$ over, $T(50)$ under $\Rightarrow$ $\boxed{\text{FALSE}}$

(c) $T(50)$ under $\Rightarrow$ $\boxed{\text{FALSE}}$

(d) $L(50)$, $L(100)$ both under, $L(100)$ closer $\Rightarrow$ $\boxed{\text{TRUE}}$

(e) $f(1) = \pi/4$, $\int_0^1 \tan^{-1}(x)\,dx < \frac{\pi}{4}\cdot 1$ $\boxed{\text{FALSE}}$

9 $P(0) = 200$, $P(t) = Ae^{kt}$, $P(0) = A = 200$

So $P(t) = 200e^{kt}$

$P(2) = 200e^{2k} = 800 \Rightarrow e^{2k} = 4$

(a) So $2k = \ln 4 = 2\ln 2$

$\Rightarrow \boxed{k = \ln 2} \Rightarrow P(t) = 200(e^{\ln 2})^t$

or $\boxed{P(t) = 200 \cdot 2^t}$

$P(5) = 200 \cdot 32 = 6400$ toads.

(b) $P = 200 \cdot 2^t$ Area $A = \pi r^2$

Initial density $\boxed{\frac{200 \text{ TOADS}}{\pi \text{ mi}^2}}$

After 5 years, Density $= \frac{6400 \text{ TOADS}}{\pi r^2} = \frac{200 \text{ TOADS}}{\pi \text{ mi}^2}$

$\Rightarrow 200\pi r^2 = 6400\pi \text{ mi}^2$

$\Rightarrow r^2 = 32 \text{ mi}^2 \Rightarrow \boxed{r = 4\sqrt{2} \text{ mi}}$

(c) $\frac{dP}{dt} = 200 \cdot 2^t \cdot \ln 2$ $\left.\frac{dP}{dt}\right|_{t=5} = 6400\ln 2$

$\frac{P}{A} = \frac{P}{\pi r^2} = \frac{200}{\pi}$ (constant)

So $P = 200r^2$

$\frac{dP}{dt} = 400r \cdot \frac{dr}{dt}$

$64\cancel{00}\ln 2 = 4\cancel{00} \cdot 4\sqrt{2} \cdot \frac{dr}{dt}$ $\Rightarrow \boxed{\left.\frac{dr}{dt}\right|_{t=5} = \frac{4\ln 2}{\sqrt{2}}}$

10 (a) $f(x) = e^{\tan^{-1}(\ln x)}$

$f'(x) = e^{\tan^{-1}(\ln x)}\left[\frac{(1/x)}{1+(\ln x)^2}\right]$

(b) $x^2y^3 + 3xe^y - 4\ln x = 5$

$x^2 \cdot 3y^2\frac{dy}{dx} + 2xy^3 + 3xe^y\frac{dy}{dx} + 3e^y - \frac{4}{x} = 0$

$(3x^2y^2 + 3xe^y)\frac{dy}{dx} = \frac{4}{x} - 2xy^3 - 3e^y$

$\frac{dy}{dx} = \frac{\frac{4}{x} - 2xy^3 - 3e^y}{3x^2y^2 + 3xe^y}$

(c) $y = \int_0^{\ln(40)} \sin(e^x)\,dx = $ constant

So $\frac{dy}{dx} = 0$.

(d) $f(x) = [x^2e^{3x} + \ln x]^5$

$f'(x) = 5[x^2e^{3x} + \ln x]^4\left[x^2e^{3x}\cdot 3 + 2xe^{3x} + \frac{1}{x}\right]$

$= 5[x^2e^{3x} + \ln x]^4\left[3x^2e^{3x} + 2xe^{3x} + \frac{1}{x}\right]$

(e) WRITE $y = x^{\ln x}$

$\ln y = \ln(x^{\ln x}) = (\ln x)^2$

$\frac{1}{y}\frac{dy}{dx} = 2(\ln x)\cdot\frac{1}{x}$

So $\frac{dy}{dx} = \left(\frac{2\ln x}{x}\right)x^{\ln x}$

(f) $f(x) = 7^{-4x}$

$f'(x) = (7^{-4x})(\ln 7)(-4)$

(g) $f(x) = \frac{e^{2x}-1}{e^{2x}+1}$

$f'(x) = \frac{(e^{2x}+1)(2e^{2x}) - (e^{2x}-1)(2e^{2x})}{(e^{2x}+1)^2}$

$= \frac{4e^{2x}}{(e^{2x}+1)^2}$

$f'(0) = \frac{4}{2^2} = \boxed{1}$

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height $= 3 + x$

radius $= r$

$x^2 + r^2 = 3^2 = 9$

$V = \frac{1}{3}\pi r^2 h$

$V(x) = \frac{1}{3}\pi(9 - x^2)(3+x)$

$V(x) = \frac{1}{3}\pi\left[27 + 9x - 3x^2 - x^3\right]$

$\frac{dV}{dx} = \frac{\pi}{3}\left[9 - 6x - 3x^2\right] = 0$

$\pi(3 - 2x - x^2) = 0$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$\boxed{x = 1}$ CRITICAL POINT

Boundary: $x = -3 \Rightarrow V = 0$

$x = +3 \Rightarrow V = 0.$

$\boxed{V(1) = \frac{\pi}{3}(8)(4) = 32\pi/3 \quad \text{RADIUS} = 2\sqrt{2} \quad \text{height} = 4}$

12 ⓐ $p^{(20)}(x) = 3\cos x$

$\boxed{p^{(21)}(x) = -3\sin x}$

ⓑ $u = 3e^{2x}$

$u' = 3 \cdot 2^1 e^{2x}$

$u'' = 3 \cdot 2^2 e^{2x}$, etc

$\cdots \Rightarrow \boxed{u^{(21)} = 3 \cdot 2^{21} e^{2x}}$

ⓒ $\boxed{f^{(21)} = 0}$ since all derivs beyond 12^{th} will vanish.

13 $a(t) = -\frac{1}{(t+1)^2} = \frac{dv}{dt}$

$v(t) = +\frac{1}{t+1} + C$

$v(0) = 1 + C = 1 \Rightarrow C = 0$

So $v(t) = \frac{1}{t+1} = \frac{dy}{dt}$

$y(t) = \ln(t+1) + C'$

$y(0) = C' = 1 \Rightarrow \boxed{y = \ln(t+1) + 1}$

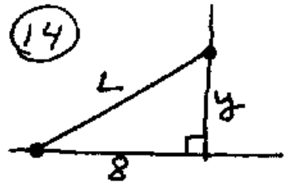
monkey reaches $y = 30$ when $30 = \ln(t+1) + 1$

$\ln(t+1) = 29$

$t + 1 = e^{29}$

$\boxed{t = e^{29} - 1}$

(NOT IN MONKEY'S LIFETIME!)

14 

$6y + y^2 = L^2$

$\cancel{2}y\frac{dy}{dt} = \cancel{2}L\frac{dL}{dt}$

when $y = 6 \Rightarrow L = 10$ ft.

If $\frac{dy}{dt} = 4$ ft/sec $\Rightarrow 6 \cdot 4 = 10\frac{dL}{dt}$

$\Rightarrow \frac{dL}{dt} = \frac{24}{10} = \boxed{2.4 \text{ ft/sec}}$

15 ⓐ $\int_{e^2}^{e^3}\left(\frac{3}{x} + \frac{2x}{e^6 - e^4}\right)dx = \left[3\ln|x| + \frac{x^2}{e^6 - e^4}\right]_{x=e^2}^{x=e^3}$

$= \left(3\ln(e^3) + \frac{e^6}{e^6 - e^4}\right) - \left(3\ln(e^2) + \frac{e^4}{e^6 - e^4}\right)$

$= 9 - 6 + \frac{e^6 - e^4}{e^6 - e^4} = 3 + 1 = \boxed{4}$

ⓑ $\int \frac{2x + 3}{\sqrt{4 - x^2}}dx = \int \frac{2x\,dx}{\sqrt{4 - x^2}} + \int \frac{3\,dx}{\sqrt{4 - x^2}}$

1st: $u = 4 - x^2$, $-du = +2x\,dx$ $\Rightarrow \int \frac{-du}{u^{1/2}} = -\int u^{-1/2}du = -\frac{2u^{1/2}}{1}$

$= -2\sqrt{4 - x^2}$.

2nd: $= \frac{1}{2}\int \frac{3\,dx}{\sqrt{1 - (\frac{x}{2})^2}} = 3\int \frac{du}{\sqrt{1 - u^2}} = 3\sin^{-1}u$

$u = \frac{x}{2}$ $dx = 2du$ $= 3\sin^{-1}\left(\frac{x}{2}\right)$.

So $\int \frac{2x + 3}{\sqrt{4 - x^2}}dx = -2\sqrt{4 - x^2} + 3\sin^{-1}\left(\frac{x}{2}\right) + C$

ⓒ $\int_0^{\sqrt{\pi}} x\sin(x^2)dx = \frac{1}{2}\int_0^{\pi}\sin u\,du = -\frac{1}{2}\cos u\Big]_0^{\pi}$

$u = x^2$ $du = 2x\,dx$ $= -\frac{1}{2}(-1 - 1) = \boxed{1}$

ⓓ $= \int e^{-3t}dt = -\frac{1}{3}e^{-3t} + C$

ⓔ $= \int_1^2 (x + x^{-2})dx = \left[\frac{1}{2}x^2 - \frac{1}{x}\right]_{x=1}^{x=2} = (2 - \frac{1}{2}) - (\frac{1}{2} - 1) = \boxed{2}$

ⓕ $\int_1^2 \frac{\ln x}{x}dx = \int_0^{\ln 2} u\,du = \frac{1}{2}u^2\Big]_0^{\ln 2} = \frac{1}{2}(\ln 2)^2$

$u = \ln x$ $du = \frac{dx}{x}$

ⓖ $u = \sin 3x$ $du = 3\cos 3x\,dx$

$\Rightarrow \frac{1}{3}\int u^2 du = \frac{1}{9}u^3 + C$

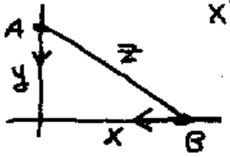
$= \frac{1}{9}\sin^3 3x + C$

ⓗ $= \frac{1}{4}\int \frac{dx}{1 + (\frac{3x}{2})^2}$ $u = \frac{3x}{2}$ $dx = \frac{2}{3}du$

$= \frac{1}{6}\int \frac{du}{1 + u^2}$

$= \frac{1}{6}\tan^{-1}u + C = \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + C$

(16)



$x^2+y^2=z^2$

$\frac{dx}{dt}=-520\ \frac{mi}{hr}$

$\frac{dy}{dt}=-520\ \frac{mi}{hr}$

$\cancel{2}x\frac{dx}{dt}+\cancel{2}y\frac{dy}{dt}=\cancel{2}z\frac{dz}{dt}$

when $y=5$ and $x=12$, then $z=13$.

$\Rightarrow (12)(-520)+(5)(-520)=13\frac{dz}{dt}$

$\Rightarrow \frac{dz}{dt}=-\frac{17}{13}(520)=-680\ \text{mi/hr}$

(17) $\int_0^x f(t)\,dt=3x^2+e^x-\cos x$

$\frac{d}{dx}\int_0^x f(t)\,dt=f(x)$.

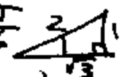
So $f(x)=6x+e^x+\sin x$

$\therefore\ f(2)=12+e^2+\sin 2$

(18) $f(x)=\cos x$

$f'(x)=-\sin x$

$30^\circ=\frac{\pi}{6}$



$f(x)\cong f\left(\frac{\pi}{6}\right)+f'\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right)$

$f\left(\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}\cong 0.866$

$f'\left(\frac{\pi}{6}\right)=-\sin\left(\frac{\pi}{6}\right)=-\frac{1}{2}=-0.5$

$x=31.8^\circ=\frac{31.8}{180}\pi$ radians

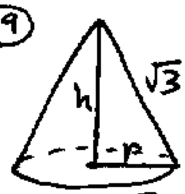
$\cos(31.8^\circ)\cong 0.866-\frac{1}{2}\left(\frac{1.8}{180}\right)\pi$

$\cong 0.866-\frac{3.1416}{200}$

$=0.866-.015708\cong\boxed{0.850}$

(Actual value $=0.84989\ldots$)

(19)



$r^2+h^2=3$

$V=\frac{1}{3}\pi r^2h$

$V(h)=\frac{1}{3}\pi(3-h^2)h$

$=\frac{1}{3}\pi(3h-h^3)$.

$0\le h\le\sqrt{3}$

$V'(h)=\frac{1}{3}\pi(3-3h^2)=\pi(1-h^2)=0$

$\Rightarrow\boxed{h=1}\ \boxed{r=\sqrt{2}}$

$V=\frac{1}{3}\pi\cdot 2\cdot 1=\boxed{2\pi/3=V}$

(endpts give $V=0$)

(20) $f(x)=e^{2x}-8e^x+12=(e^x-6)(e^x-2)$

$f'(x)=2e^{2x}-8e^x=2e^x(e^x-4)$

$f''(x)=4e^{2x}-8e^x=4e^x(e^x-2)$

y-intercept: $f(0)=1-8+12=5$ $\boxed{(0,5)}$

x-intercepts: $(e^x-6)(e^x-2)=0$

$e^x=6$ or $e^x=2$

$x=\ln 6$ or $x=\ln 2$

$\boxed{(\ln 2,0)\quad(\ln 6,0)}$

CRIT PTS: $2e^x(e^x-4)=0$

$\Rightarrow e^x=4\quad x=\ln 4$

$\Rightarrow\boxed{(\ln 4,-4)\ \text{REL. MIN}}$

$f(\ln 4)=(4-6)(4-2)=-4$

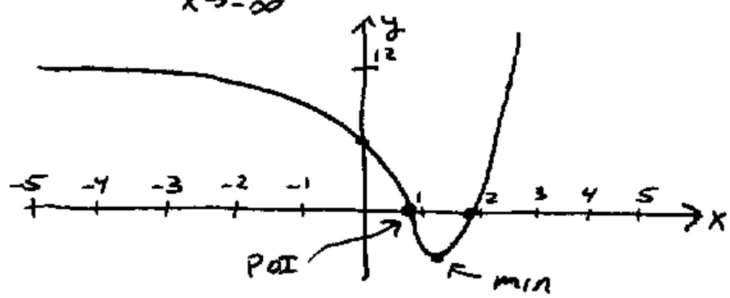
$f''(\ln 4)=4\cdot 4(2)=32>0$

POI: $4e^x(e^x-2)=0$

$\Rightarrow e^x=2\quad x=\ln 2$

$\boxed{(\ln 2,0)\ \text{is a pt. of inflection}}$

Asymptotes: $\lim_{x\to-\infty}f(x)=(-6)(-2)=12$



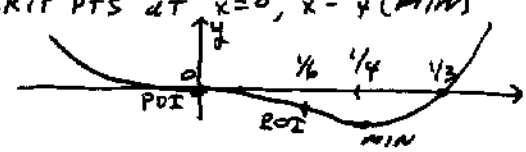
(21) $f(x)=x^4-\frac{1}{3}x^3$

$f'(x)=4x^3-x^2$ $\quad f''(x)=12x^2-2x$

$f(x)$: x-int at $x=0$, $x=\frac{1}{3}$ $\quad$ POI at $x=0$, $x=1/6$

CRIT PTS at $x=0$, $x=\frac{1}{4}$ (MIN)

(NOT TO SCALE)



$f'(x)$: x-int. at $x=0$, $x=\frac{1}{4}$

CRIT PTS AT $x=0$, $x=1/6$

$f'''(x)=24x-2\Rightarrow$ POI AT $x=\frac{1}{12}$

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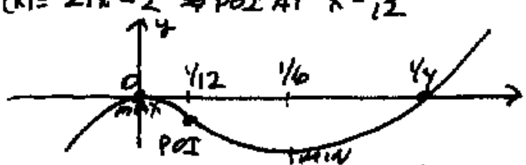
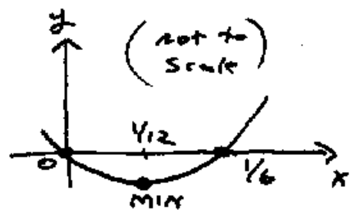


$f''(x)$: x-int. at $x=0$, $x=1/6$

CRIT PT AT $x=1/12$

$f''''(x)=24\neq 0$ NO POI.

(not to scale)



(22) $y = \dfrac{x^2 - x + 1}{x-1} = x + \dfrac{1}{x-1}$

$\dfrac{dy}{dx} = 1 - \dfrac{1}{(x-1)^2}$

$\dfrac{d^2y}{dx^2} = + \dfrac{2}{(x-1)^3}$

y-intercept at $(0,-1)$

no x-intercepts.

CRIT PT where $\dfrac{1}{(x-1)^2} = 1$

$\Rightarrow (x-1)^2 = 1$

$x - 1 = \pm 1 \Rightarrow x = 0,\ x = 2$

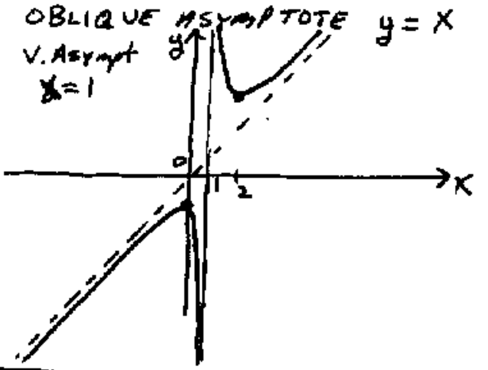
$f''(0) = -2 < 0 \Rightarrow$ LOCAL MAX $(0,-1)$

$f''(2) = +2 > 0 \Rightarrow$ LOCAL MIN $(2,3)$

NO PTS. OF INFLECTION

OBLIQUE ASYMPTOTE $y = x$

V. Asympt $x = 1$



(23) $f(x) = \dfrac{x-1}{x^2} = \dfrac{1}{x} - \dfrac{1}{x^2}$

$f'(x) = -\dfrac{1}{x^2} + \dfrac{2}{x^3} = \dfrac{2-x}{x^3}$

$f''(x) = +\dfrac{2}{x^3} - \dfrac{6}{x^4} = \dfrac{2x-6}{x^4}$

$x = 0$ IS A vertical asymptote

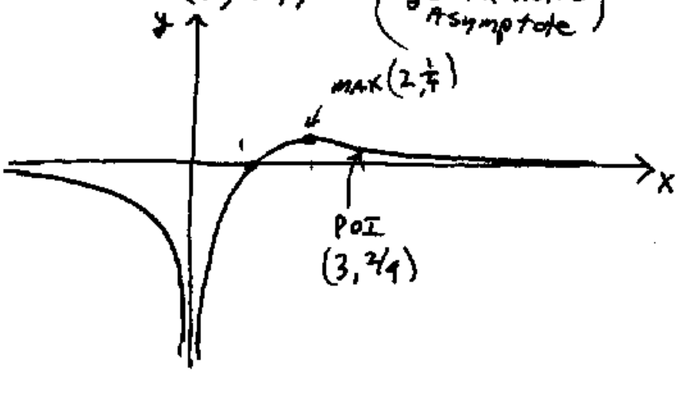
x-intercept at $(1,0)$

CRIT PT AT $x = 2$

$f''(2) < 0 \Rightarrow$ LOCAL MAX $\left(2, \frac{1}{4}\right)$

P.O.I. when $2x - 6 = 0 \Rightarrow x = 3$

$(3, 2/9)$ ($y = 0$ a horiz Asymptote)



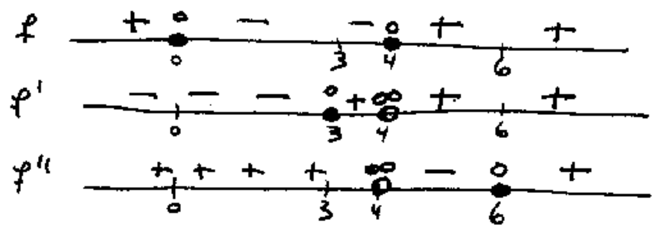
(24) $f(x) = x\sqrt[3]{x-4}$ $\quad x=0,\ x=4$ x-intercepts.

$f'(x) = \dfrac{4(x-3)}{3(x-4)^{2/3}}$ $\quad x=3$ CRIT POINT

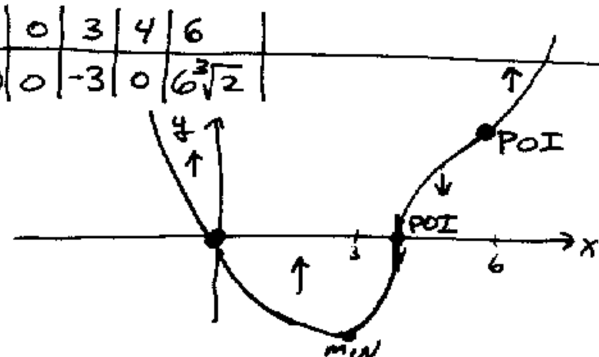
$x = 4$ C.P., vert. tangent

$f''(x) = \dfrac{4(x-6)}{9(x-4)^{5/3}}$ $\quad x=6$ PT. of Inflect.

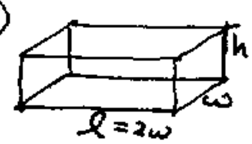
$x = 4$ Also



x	0	3	4	6
f(x)	0	-3	0	$6\sqrt[3]{2}$



(25)



$V = lwh = \frac{10}{3}$

$l = 2w$

$\Rightarrow 2w^2h = \frac{10}{3}$

Cost $C = [2\cdot(2wh) + 2\cdot wh + 2w\cdot w](.2) + 2w^2(.3)$

$C = (6wh + 2w^2)(.2) + .6w^2$

$C = 1.2wh + w^2$ and $h = \dfrac{10}{6w^2} = \dfrac{5}{3w^2}$

$C(w) = 1.2w\left(\dfrac{5}{3w^2}\right) + w^2$

$C(w) = \dfrac{2}{w} + w^2$

$C'(w) = -\dfrac{2}{w^2} + 2w = 0 \Rightarrow 2w = \dfrac{2}{w^2}$

$2w^3 = 2$

$\boxed{w = 1}$

$\therefore \boxed{l = 2}\ \boxed{h = \frac{5}{3}}$ ($C''(1) > 0 \Rightarrow$ MINIMUM)

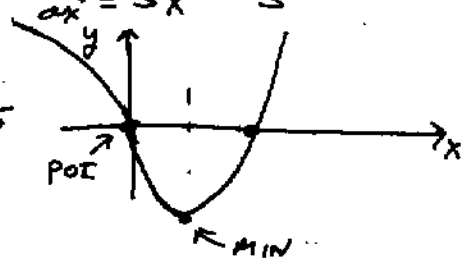
COST $= 2 + 1 = \$3.00$ per box.

(26) $y = 3x^{5/3} - 5x$ $\quad \dfrac{dy}{dx} = 5x^{2/3} - 5$

$\dfrac{d^2y}{dx^2} = \dfrac{10}{3x^{1/3}}$

$y = 0 \Rightarrow x = 0$ OR $3x^{2/3} = 5$

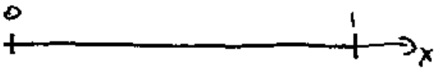
$\Rightarrow x = \left(\frac{5}{3}\right)^{3/2}$



$\frac{dy}{dx} = 0 \Rightarrow x = 1$ $(1, -2)$

POI at $x = 0$

27



$accel = a = \text{constant}$

$v(t) = at + C$

$v(0) = 0 + C = 0$ (rest)

$\therefore v(t) = at = \frac{dx}{dt}$

$x(t) = \frac{1}{2}at^2 + C'$

$x(0) = 0 + C' = 0$

so $x(t) = \frac{1}{2}at^2$

If time on runway $= T$,

then $v(T) = aT = 100\ ^{mi}/_{hr}$

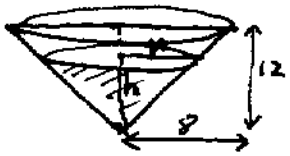
$x(T) = \frac{1}{2}aT^2 = 1$

$\Rightarrow \frac{1}{2}T(aT) = \frac{1}{2}T(100) = 1$

$\Rightarrow \boxed{T = \frac{1}{50}\text{ hr}}$

and $\boxed{a = \frac{100}{T} = 5000\ ^{mi}/_{hr^2}}$

28



$\frac{r}{h} = \frac{12}{8} = \frac{3}{2}$

$3h = 2r$

$r = \frac{3}{2}h$

$V = \frac{1}{3}\pi r^2 h = \frac{3}{4}\pi h^3$

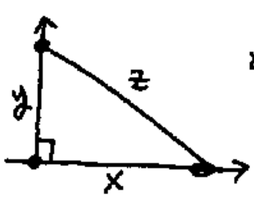
$\frac{dV}{dt} = \frac{9}{4}\pi h^2 \frac{dh}{dt}$

$\frac{dV}{dt} = -36\ ^{ft^3}/_{min}$, $h = 9\text{ft}$

$\Rightarrow -36 = \frac{9}{4}\pi \cdot 81 \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{-36\cdot 4}{9\cdot 81\pi} = \boxed{-\frac{16}{81\pi}\ ^{ft}/_{min}}$

29



$x^2 + y^2 = z^2$

$\frac{dx}{dt} = 50\ ^{km}/_{hr}$

$\frac{dy}{dt} = 10\ ^{km}/_{hr}$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$\frac{dz}{dt} = \frac{50x + 10y}{z}\ ^{km}/_{hr}$

we can compute this with add'l info.

For example after 1 hour, $x = 50$, $y = 10$

and $z = 10\sqrt{26}$

$\Rightarrow \frac{dz}{dt} = \frac{2500 + 100}{10\sqrt{26}} = 10\sqrt{26}\ ^{km}/_{hr}$.

But, RATIOS $\frac{x}{z} = \frac{5}{\sqrt{26}}$ and $\frac{y}{z} = \frac{1}{\sqrt{26}}$

REMAIN CONSTANT, so $\frac{dz}{dt} = 10\sqrt{26}\ ^{km}/_{hr}$ always.

30 $v_0 = 200\ ^{ft}/_{sec}$ $\quad a = -10\ ^{ft}/_{sec^2}$

$v(t) = -10t + 200 \qquad x_0 = 0$

$x(t) = -5t^2 + 200t$

when $x = 1280$, we have

$-5t^2 + 200t = 1280$

$5t^2 - 200t + 1280 = 0$

$t = \frac{200 \pm \sqrt{40000 - 25600}}{10}$

$= \frac{200 \pm 120}{10} = 32\text{ sec}$ or 8 sec

$v(8) = -80 + 200 = 120\ ^{ft}/_{sec}$ which exceeds the speed limit.

31 (a) $\lim_{x\to\infty} 2x\ln\left(1+\frac{3}{x}\right) \to \infty\cdot 0$

$= \lim_{x\to+\infty}\left[\frac{2\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}}\right] \to \frac{0}{0}$

$\therefore = \lim_{x\to+\infty} \frac{2\left(\frac{-3/x^2}{1+3/x}\right)}{\left(-1/x^2\right)} = \lim_{x\to+\infty}\left(\frac{6}{1+3/x}\right) = \boxed{6}$

(b) $\lim_{x\to 0}\left(\frac{xe^x - x}{1-\cos x}\right) \to \frac{0}{0}$ $\therefore = \lim_{x\to 0}\left(\frac{xe^x + e^x - 1}{\sin x}\right) \to \frac{0}{0}$

$\therefore = \lim_{x\to 0}\left(\frac{xe^x + 2e^x}{\cos x}\right) = \frac{2}{1} = \boxed{2}$

(c) $L = \lim_{x\to 0}(1+\sin 2x)^{1/x} \to 1^\infty$ indeterminate

$\ln L = \lim_{x\to 0}\left[\frac{\ln(1+\sin 2x)}{x}\right] \to \frac{0}{0}$ $\quad \ln L = 2$

$\therefore = \lim_{x\to 0}\left[\frac{2\cos 2x}{1+\sin 2x}\right] = \frac{2}{1} = 2$ $\quad$ so $L = \boxed{e^2}$

(d) $\lim_{x\to 0}\left(\frac{\sec x - 1}{x^2}\right) \to \frac{0}{0}$ $\therefore = \lim_{x\to 0}\left(\frac{\sec x\tan x}{2x}\right)$

$= \lim_{x\to 0}\left[\frac{\sin x}{2(\cos x)^2\cdot x}\right] = \lim_{x\to 0}\left(\frac{1}{2(\cos x)^2}\right)\cdot\lim_{x\to 0}\left(\frac{\sin x}{x}\right)$

$= \frac{1}{2}\cdot 1 = \boxed{\frac{1}{2}}$

(e) For x near 0, $\sin x \cong x$, $\sin(x^2) \cong x^2$

so $\frac{x\sin x}{\sin(x^2)} \cong \frac{x\cdot x}{x^2} = 1$ $\therefore \lim_{x\to 0}\frac{x\sin x}{\sin(x^2)} = 1$

(f) $\lim_{x\to 0}\left(\frac{xe^x - x}{e^x - x - 1}\right) \to \frac{0}{0}$ $\therefore = \lim_{x\to 0}\left(\frac{xe^x + e^x - 1}{e^x - 1}\right) \to \frac{0}{0}$

$\therefore = \lim_{x\to 0}\left(\frac{xe^x + 2e^x}{e^x}\right) = \frac{2}{1} = \boxed{2}$

32 (a) $L = \lim_{x\to\infty} x^{2/x} \to \infty^0$ (Indet.)

$\ln L = \lim_{x\to\infty} \ln\left(x^{2/x}\right) = \lim_{x\to\infty}\left(\frac{2\ln x}{x}\right) \to \frac{\infty}{\infty}$

$\therefore = \lim_{x\to\infty}\left(\frac{2/x}{1}\right) = 0$ $\quad \ln L = 0$

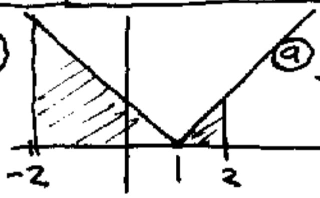
So $L = e^0 = \boxed{1}$

(b) Write $y = x^{2/x}$ use Log. diff'n.

$\ln y = \ln x^{2/x} = \frac{2\ln x}{x}$

$\frac{1}{y}\frac{dy}{dx} = \frac{x\cdot\frac{2}{x} - 2\ln x}{x^2} = \frac{2-2\ln x}{x^2}$

So $\boxed{\frac{dy}{dx} = \frac{2(1-\ln x)}{x^2}x^{2/x}}$

33 

(a) $\int_{-2}^{2}|x-2|\,dx$ = area

$= \frac{1}{2}(3)(3) + \frac{1}{2}(1)(1)$

$= \frac{9}{2} + \frac{1}{2} = \boxed{5}$

(b) $\bar{f} = \frac{1}{5}\int_0^5 |x-2|\,dx = \frac{1}{5}\left[\frac{1}{2}(1\cdot1) + \frac{1}{2}(4\cdot4)\right]$

$= \frac{1}{5}\cdot\frac{17}{2} = \frac{17}{10} = \boxed{1.7}$

34 $v_0 = 25$ m/s $\quad a = -5$ m/s²

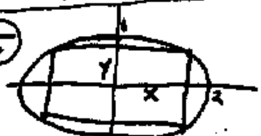
$v(t) = -5t + 25 \qquad x_0 = 0$

$x(t) = -\frac{5}{2}t^2 + 25t$

STOPPED when $v = 0 \Rightarrow -5t + 25 = 0$

so $\boxed{t = 5}$

$x(5) = -\frac{5}{2}(25) + 125 = \frac{125}{2} = \boxed{62\frac{1}{2}\text{ Meters}}$

35 

$A = (2x)(2y) = 4xy$

$\frac{x^2}{4} + y^2 = 1$

$x = 2(1-y^2)^{\frac{1}{2}}$

So $A(y) = 8y(1-y^2)^{\frac{1}{2}} \qquad 0 \le y \le 1$

$A'(y) = (8y)\frac{1}{2}(1-y^2)^{-1/2}(-2y) + 8(1-y^2)^{1/2}$

$= \frac{8(1-y^2) - 8y^2}{\sqrt{1-y^2}} = \frac{8-16y^2}{\sqrt{1-y^2}} = 0$

$\Rightarrow 16y^2 = 8 \Rightarrow y^2 = \frac{1}{2}$

$y = \frac{\sqrt{2}}{2} \Rightarrow x = \sqrt{2}$

$\Rightarrow$ $\boxed{\text{Dimensions are } 2\sqrt{2} \text{ by } \sqrt{2}}$

36 $\int_{-3}^{2} f(x)\,dx = \int_{-3}^{0}(x+3)\,dx + \int_0^2 x^2\,dx$

$= \left[\frac{x^2}{2} + 3x\right]_{-3}^{0} + \left[\frac{x^3}{3}\right]_0^2$

$= 0 - \left(\frac{9}{2} - 9\right) + \frac{8}{3} - 0$

$= \frac{9}{2} + \frac{8}{3} = \boxed{\frac{43}{6}}$

37 $n = 2 \Rightarrow \Delta x = 1$

x	0	.5	1	1.5	2
f(x)	1	1.118	1.414	1.803	2.236

(a) $L = 1(f(0) + f(1)) = \underline{2.414}$

$R = 1(f(1) + f(2)) = 1.414 + 2.236 = \underline{3.650}$

$T = \frac{L+R}{2} = \underline{3.032}$

$M = 1(f(.5) + f(1.5)) = \underline{2.921}$

(with $n = 4$) $S = \frac{2M+T}{3} = \underline{2.958}$

(b) $L < M < S < T < R$

(c) On $[0,2]$, $f'(x) > 0 \Rightarrow$ Increasing.

$f''(x) > 0 \Rightarrow$ concave up. 

Hence L underestimates, R overestimates, and T overestimates.

38 $a = 30\sin t$ ft/sec² $\quad v_0 = 30$ ft/sec

$\Rightarrow v(t) = -30\cos t + C$

$v(0) = -30 + C = 30 \Rightarrow C = 60$

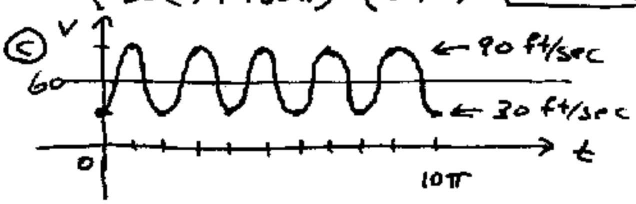
(a) So $\boxed{v(t) = -30\cos t + 60}$

(b) $x(3\pi) - x(0) = \int_0^{3\pi} x'(t)\,dt = \int_0^{3\pi} v(t)\,dt$

$= \int_0^{3\pi}(-30\cos t + 60)\,dt$

$= -30\sin t + 60t\Big]_0^{3\pi}$

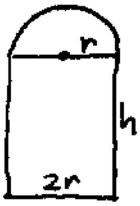
$= (-30(0) + 180\pi) - (0+0) = \boxed{180\pi \text{ ft}}$

(c) 

CAR'S speed is varying wildly.

(d) CAR IS ALSO PERIODICALLY ABOVE THE SPEED LIMIT.

(39)



$P = 2h + 2r + \pi r = 24$

$2h + (2+\pi)r = 24$

MAXIMIZE AREA

$A = 2rh + \frac{1}{2}\pi r^2$

Solve for $h = \frac{24-(2+\pi)r}{2}$

$h = 12 - (1+\frac{\pi}{2})r$

$A(r) = 24r - (2+\pi)r^2 + \frac{\pi}{2}r^2$

$A(r) = 24r - (2+\frac{\pi}{2})r^2$

endpts: $r=0$ and $h=0$

$\Rightarrow r = \frac{24}{2+\pi}$

CRIT PT: $A'(r) = 24 - (4+\pi)r = 0$

$\Rightarrow r = \frac{24}{4+\pi}$

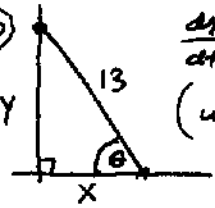
$A(0) = 0$

$\boxed{A\left(\frac{24}{4+\pi}\right) = \frac{576}{4+\pi} - \left(\frac{4+\pi}{2}\right)\left(\frac{24}{4+\pi}\right)^2 = \frac{288}{4+\pi} \approx 40}$

$A\left(\frac{24}{2+\pi}\right) = \frac{\pi}{2}\left(\frac{24}{2+\pi}\right)^2 = \frac{288\pi}{(2+\pi)^2} \approx 34$

MAX area at CRITICAL POINT.

(40)



$\frac{dx}{dt} = .1$ ft/sec

(when $y = 12 \Rightarrow x = 5$)

$\cos\theta = \frac{x}{13}$

$-\sin\theta \frac{d\theta}{dt} = \frac{1}{13}\frac{dx}{dt}$

when $y = 12, x = 5 \Rightarrow \sin\theta = \frac{12}{13}$

$\Rightarrow -\frac{12}{13}\frac{d\theta}{dt} = \frac{1}{13}(.1)$

$\Rightarrow \frac{d\theta}{dt} = \frac{-.1}{12} = \boxed{-\frac{1}{120} \text{ rad/sec}}$

(41) $s(t) = -2t^3 - 3t^2 + 12t + 40$

$v(t) = -6t^2 - 6t + 12$

$= -6(t^2 + t - 2)$ $\quad 0 \le t \le 3$

$v = 0$ when $(t+2)(t-1) = 0 \Rightarrow t = 1$

$v(0) = 12$ (forward) $\quad v(1) = 0$ (stop)

$v(3) = -54 - 18 + 12 < 0$ (backward)

$s(0) = 40 \quad s(1) = -2 - 3 + 12 + 40 = 47$

$s(3) = -54 - 27 + 36 + 40 = -5$

TOTAL DISTANCE $= |47-40| + |-5-47|$

$= 7 + 52 = \boxed{59 \text{ units}}$

(42) $f(x) = x^3 e^{-x}$

$f'(x) = -x^3 e^{-x} + 3x^2 e^{-x}$

$= x^2 e^{-x}(3-x)$

$f''(x) = x^3 e^{-x} - 6x^2 e^{-x} + 6x e^{-x}$

$= x e^{-x}(x^2 - 6x + 6)$

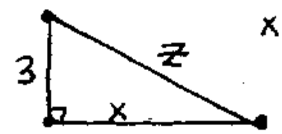
Decreasing where $f'(x) < 0 \Rightarrow \boxed{x > 3}$

Decreasing most rapidly (for $x > 3$) where

$f''(x) = 0 \Rightarrow x = \frac{6 \pm \sqrt{36-24}}{2}$

$\Rightarrow \boxed{x = 3 + \sqrt{3}}$

(43)



x = ground dist between plane and car.

$x^2 + 9 = z^2 \quad 2x\frac{dx}{dt} = 2z\frac{dz}{dt}$

when $z = 5 \Rightarrow x = 4$. $\quad \frac{dz}{dt} = -160$ mi/hr

$\Rightarrow 4\frac{dx}{dt} = (5)(-160) \quad \frac{dx}{dt} = -200$ mi/hr.

Now Adjust for ground speed of plane (120 mi/hr)

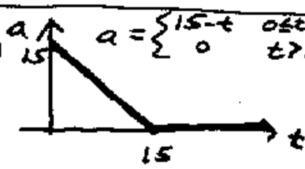
$\Rightarrow$ car's velocity $= 200 - 120 = \boxed{80 \text{ mi/hr}}$

(44) $\int_0^1 f(1-x)\,dx = -\int_1^0 f(u)\,du$

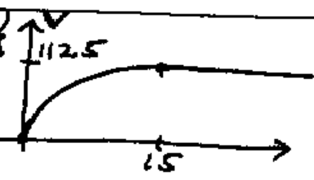
$u = 1-x \quad du = -dx \qquad = +\int_0^1 f(u)\,du$

So $\int_0^1 f(1-x)\,dx = \int_0^1 f(x)\,dx$.

(45)



$a = \begin{cases} 15-t & 0 \le t \le 15 \\ 0 & t > 15 \end{cases}$



on $[0,15]$ $v = 15t - \frac{1}{2}t^2 + C \quad v(0) = C = 0$

$\Rightarrow v(t) = 15t - \frac{1}{2}t^2 \quad v(15) = 225 - \frac{225}{2} = \frac{225}{2}$

for $t > 15$ $\quad v(t) = \frac{225}{2} = 112.5$ ft/sec.

$=$ MAX. velocity.

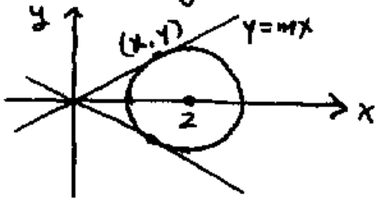
(46) $\int_{-3}^{2} f(x)\,dx = \int_{-3}^{0}(x+3)\,dx + \int_0^2 x^2\,dx$

$= \boxed{43/6}$ (SAME PROBLEM AS #36 (oops!))

(47) $\boxed{x^2-4x+y^2+3=0}$ ⓐ

$x^2-4x+4+y^2=-3+4$

$(x-2)^2+y^2=1$



LINE THROUGH $(0,0) \Rightarrow \boxed{y=mx}$ ⓑ

$2x-4+2y\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=\frac{4-2x}{2y}=\boxed{\frac{2-x}{y}=m}$ ⓒ

ⓐ and ⓑ $\Rightarrow x^2-4x+m^2x^2+3=0$

ⓑ and ⓒ $\Rightarrow m^2x=2-x$

So $x^2-4x+(2-x)x+3=0$

$x^2-4x+2x-x^2+3=0$

$3-2x=0 \Rightarrow x=\frac{3}{2}$

$\Rightarrow y^2=4x-x^2-3=6-\frac{9}{4}-3=\frac{3}{4}$

$\Rightarrow y=\pm\frac{\sqrt{3}}{2} \Rightarrow m=\pm\frac{\sqrt{3}}{3}$

$\Rightarrow$ $\boxed{\text{Lines } y=\frac{\sqrt{3}}{3}x \text{ and } y=-\frac{\sqrt{3}}{3}x}$

(48) $f(x)=2\sin x-3\cos x \quad x_0=\frac{3\pi}{2}$

$f'(x)=2\cos x+3\sin x$

$f(3\pi/2)=-2 \quad f'(3\pi/2)=-3$

$$\boxed{y=-2-3\left(x-\frac{3\pi}{2}\right)}$$

(49) $x^2y+2y^3=3x+2y+54 \quad (2,3)$

$2xy+x^2\frac{dy}{dx}+6y^2\frac{dy}{dx}=3+2\frac{dy}{dx}$

$(x^2+6y^2-2)\frac{dy}{dx}=3-2xy$

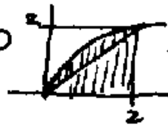
$\left.\frac{dy}{dx}\right|_{(2,3)}=\left.\frac{3-2xy}{x^2+6y^2-2}\right|_{(2,3)}=\frac{3-12}{4+54-2}=-\frac{9}{56}$

TL is $\boxed{y-3=-\frac{9}{56}(x-2)}$

(50) $f(2)=2,\ 0<f'(2)<1,\ f''(2)<0$

$\int_0^2 f'(x)\,dx=f(2)-f(0)=2-0=2$

$\int_0^2 f(x)\,dx>0$



AREA $\Delta=\frac{1}{2}(2)(2)=2$

$\Rightarrow 2<\int_0^2 f(x)\,dx<4$

So

$f''(2)<0<f'(2)<1<f(2)=\int_0^2 f'(x)\,dx<\int_0^2 f(x)\,dx$

(51)

$\frac{dx}{dt}=220$ ft/sec

$\frac{x}{100}=\tan\theta \quad \frac{1}{100}\frac{dx}{dt}=\sec^2\theta\frac{d\theta}{dt}$

$\Rightarrow \frac{d\theta}{dt}=\frac{1}{100}\cos^2\theta\frac{dx}{dt}=2.2\cos^2\theta$

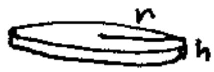
ⓐ $x=0 \Rightarrow \theta=0 \Rightarrow \frac{d\theta}{dt}=2.2$ RAD/sec

ⓑ $x=240 \Rightarrow \cos\theta=\frac{100}{\sqrt{(100)^2+(240)^2}}$

So $\frac{d\theta}{dt}=2.2\left[\frac{100^2}{(100)^2+(240)^2}\right]=\frac{22000}{67600}=\frac{55}{169}$ RAD/sec

$(\doteq .325$ RAD/sec$)$

(52)



$V=\pi r^2h=Ah$, A = area.

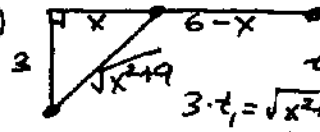
$\frac{dV}{dt}=A\frac{dh}{dt}+h\frac{dA}{dt}$

$\frac{dV}{dt}=-5$ ft³/hr, $r=500 \Rightarrow A=250000\pi$ ft², $\frac{dh}{dt}=-.001$, $h=.01$

$\Rightarrow -5=(250000\pi)(-.001)+(.01)\frac{dA}{dt}$

$\Rightarrow \frac{dA}{dt}=(25000\pi-500)$ ft²/hr AREA INCREASING.

(53)



t_1 = time in snow

t_2 = time on road.

$3\cdot t_1=\sqrt{x^2+9} \quad 4t_2=6-x$

Total time $T=t_1+t_2=\frac{1}{3}\sqrt{x^2+9}+\frac{1}{4}(6-x)$

$\frac{dT}{dx}=\frac{x}{3\sqrt{x^2+9}}-\frac{1}{4}=0 \Rightarrow x=\frac{9}{\sqrt{7}}$.

$T\left(\frac{9}{\sqrt{7}}\right)=\frac{1}{3}\sqrt{\frac{81}{7}+9}+\frac{1}{4}\left(6-\frac{9}{\sqrt{7}}\right)=\underline{2.16\text{ hrs.}}$

So, you'll get there before midnight.

(54) ⓐ $F'(x)=\sin 2x+\cos 2x \quad F(0)=0$

$\Rightarrow F(x)=-\frac{1}{2}\cos 2x+\frac{1}{2}\sin 2x+C$

$F(0)=-\frac{1}{2}+C=0 \Rightarrow C=\frac{1}{2} \Rightarrow \boxed{F(x)=-\frac{1}{2}\cos 2x+\frac{1}{2}\sin 2x+\frac{1}{2}}$

ⓑ $F'(x)=x^{1/2}-x^{3/2} \Rightarrow F(x)=\frac{2}{3}x^{3/2}-\frac{2}{5}x^{5/2}+C$

$F(0)=0+C=0 \Rightarrow C=0 \Rightarrow \boxed{F(x)=\frac{2}{3}x^{3/2}-\frac{2}{5}x^{5/2}}$

ⓒ $f'(x)=2x-3 \Rightarrow f(x)=x^2-3x+C$

$f(1)=1-3+C=C-2=0 \Rightarrow C=2 \Rightarrow \boxed{f(x)=x^2-3x+2}$

$f(x)=0 \Rightarrow x^2-3x+2=(x-2)(x-1)=0$

$\Rightarrow x=1$ or $\boxed{x=2}$