MATH 1a Exam 2 Solutions

1) $f(x) = \dfrac{x^2+4x+3}{x(x+4)} = \dfrac{(x+1)(x+3)}{x(x+4)}$

$f'(x) = \dfrac{-6x-12}{x^2(x+4)^2} = \dfrac{-6(x+2)}{x^2(x+4)^2}$

$f''(x) = \dfrac{6(3x^2+12x+16)}{x^3(x+4)^3}$

(a) $f(0)$ is undefined

$f(x)=0 \Rightarrow x=-1,\ x=-3$ are x-intercepts

(b) $f'(x)=0 \Rightarrow x=-2$ CRITICAL POINT

$f''(-2) = \dfrac{24}{-64} < 0 \Rightarrow$ LOCAL MAX.

(c) $f''(x)=0 \Rightarrow$ NO SOLUTIONS.

$\therefore$ NO POINTS OF INFLECTION

(d) $x=0$ and $x=-4$ give vertical asymptotes.

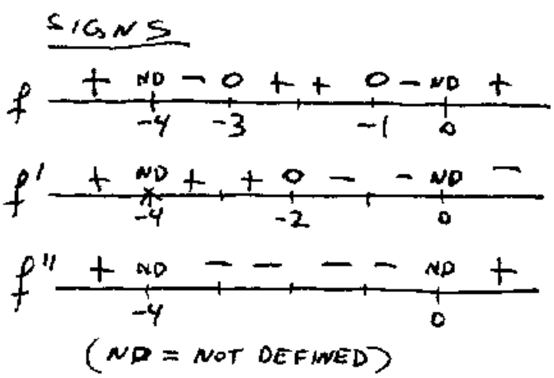
$\lim\limits_{x\to 0^+} f(x) = \dfrac{++}{++} = +\infty$ $\qquad \lim\limits_{x\to -4^+} f(x) = \dfrac{--}{-+} = -\infty$

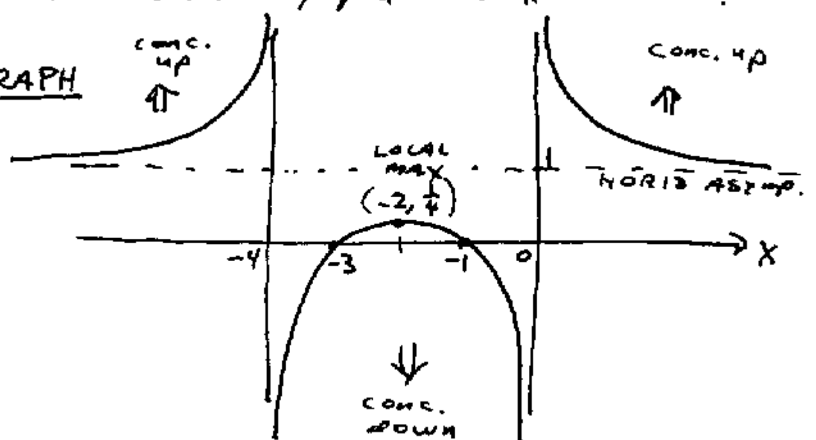
$\lim\limits_{x\to 0^-} f(x) = \dfrac{++}{-+} = -\infty$ $\qquad \lim\limits_{x\to -4^-} f(x) = \dfrac{--}{--} = +\infty$

(e) $\lim\limits_{x\to +\infty} f(x) = \lim\limits_{x\to +\infty}\left(\dfrac{x^2+4x+3}{x^2+4x}\right) \to \dfrac{\infty}{\infty} \quad \therefore = \lim\limits_{x\to +\infty} \dfrac{2x+4}{2x+4} = 1$

So $y=1$ is a horizontal asymptote as $x \to +\infty$.

Similarly, $y=1$ is also a horizontal asymptote as $x \to -\infty$.





3) (a) $\lim\limits_{x\to 1}\left(\dfrac{x^4-5x^2+3x+1}{x^3-2x+1}\right) \to \dfrac{0}{0} \quad \therefore = \lim\limits_{x\to 1}\left(\dfrac{4x^3-10x+3}{3x^2-2}\right) = \dfrac{-3}{1} = -3$

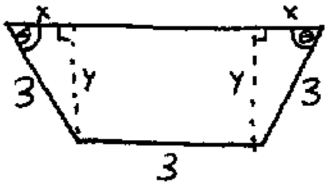
(by L'Hôpital's Rule)

(b) Newton's method says that $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

In particular $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 1 - \dfrac{(-1)}{4} = 1 + \dfrac{1}{4} = \boxed{1.25}$

$f(x) = x^4 - 2 \qquad f(1) = 1-2 = -1$

$f'(x) = 4x^3 \qquad f'(1) = 4$

3) 

$\text{Area} = 3y + 2\left(\frac{1}{2}xy\right) = 3y + xy$

we know that $x^2 + y^2 = 9$ and $\left\{\begin{array}{l} x = 3\cos\theta \\ y = 3\sin\theta \end{array}\right\}$

<u>Method I:</u> (Use y as variable) $x = \sqrt{9-y^2} \Rightarrow A(y) = 3y + y(9-y^2)^{1/2}$.

Endpoints are given by $0 \le y \le 3$.

$A'(y) = 3 + (9-y^2)^{1/2} + \frac{1}{2}y(9-y^2)^{-1/2}(-2y) = 3 + \sqrt{9-y^2} - \frac{y^2}{\sqrt{9-y^2}} = 0$

this gives us $3 = \frac{y^2 - (9-y^2)}{\sqrt{9-y^2}}$ (after transposing and finding a common denom. for RHS)

$\Rightarrow 3\sqrt{9-y^2} = 2y^2 - 9$

Squaring both sides gives $9(9-y^2) = 4y^4 - 36y^2 + 81$

So $81 - 9y^2 = 4y^4 - 36y^2 + 81 \Rightarrow 4y^4 - 27y^2 = 0$

$\Rightarrow y^2(4y^2 - 27) = 0 \Rightarrow y = 0,\ y = -\frac{3\sqrt{3}}{2},\ y = \frac{3\sqrt{3}}{2}$

$y=0$ gives $A=0$ (endpoint), $y = -\frac{3\sqrt{3}}{2}$ is not in domain,

and other endpoint $y=3$ gives $A = 9$. The critical point $y = \frac{3\sqrt{3}}{2}$

gives $A = \frac{9\sqrt{3}}{2} + \frac{3\sqrt{3}}{2}\left(9 - \frac{27}{4}\right)^{1/2} = \frac{9\sqrt{3}}{2} + \frac{3\sqrt{3}}{2}\cdot\frac{3}{2} = \frac{27\sqrt{3}}{4} > \frac{27}{4}\cdot\frac{3}{2} = \frac{81}{8} > 10$

So the CRITICAL POINT $y = \frac{3\sqrt{3}}{2}$, $x = \frac{3}{2}$ gives MAXIMUM AREA.

<u>Note:</u> Using x as variable would proceed similarly.

<u>Method II:</u> (Use θ as variable) $A(\theta) = 9\sin\theta + 9\sin\theta\cos\theta$

$A'(\theta) = 9\cos\theta + 9\cos^2\theta - 9\sin^2\theta = 0$. $\quad 0 \le \theta \le \pi/2$ (in practical terms)

Use fact that $\sin^2\theta = 1 - \cos^2\theta$ to get

$9\cos\theta + 9\cos^2\theta - 9(1-\cos^2\theta) = 0$

$18\cos^2\theta + 9\cos\theta - 9 = 0 \Rightarrow 2\cos^2\theta + \cos\theta - 1 = 0$

By quadratic formula, solve for $\cos\theta = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4}$.

So $\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$ (out of domain.)

$\cos\theta = \frac{1}{2}$ gives $\theta = 60° = \pi/3$ radians $\Rightarrow \text{Area} = 9\frac{\sqrt{3}}{2} + 9\frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{27\sqrt{3}}{4}$

<u>Endpoints:</u> $\theta = 0 \Rightarrow \text{Area} = 0 \quad \theta = \pi/2 \Rightarrow \text{Area} = 9$

So MAXIMUM AREA occurs where $\theta = 60° = \pi/3$ radians.

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4) $f(x) = xe^{-2x}$ passes through $(0,0)$

$f'(x) = e^{-2x}(1-2x)$ CRIT PT AT $x = \frac{1}{2}$

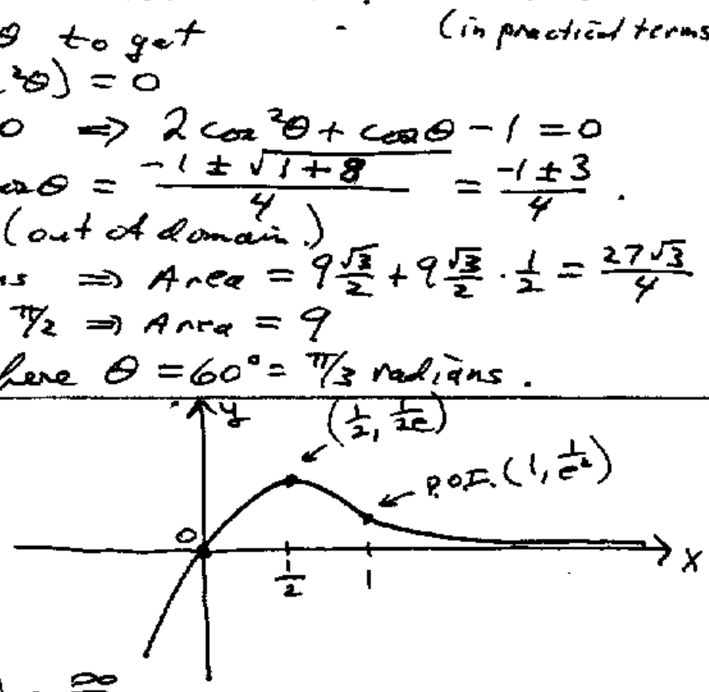
$f''(x) = 4e^{-2x}(x-1)$ P.O.I. at $x=1$

$f''\left(\frac{1}{2}\right) = \frac{4}{e}\left(-\frac{1}{2}\right) < 0 \Rightarrow$ LOCAL MAX

$f\left(\frac{1}{2}\right) = \frac{1}{2}e^{-1} = \frac{1}{2e}$

$f(1) = e^{-2} = \frac{1}{e^2}$

$f'(0) = 1$ (slope at $(0,0)$)



$\lim_{x\to+\infty}\left(\frac{x}{e^{2x}}\right) \to \frac{\infty}{\infty}$

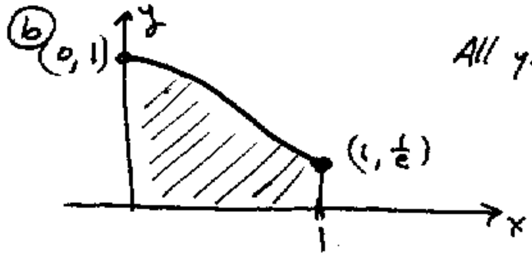
$\therefore = \lim_{x\to+\infty} \frac{1}{2e^{2x}} = 0 \quad y=0$ is HORIZ. ASYMPTOTE.

5) a) $g(x) = x^4 - 2x^2 - x + 5$ on interval $[0, 2]$.

note that $g(0) = 5$ and $g(2) = 16 - 8 - 2 + 5 = 11$, so $\dfrac{g(2)-g(0)}{2-0} = \dfrac{11-5}{2} = 3$.

By the MEAN VALUE THEOREM (since $g(x)$ is continuous and differentiable on $[0,2]$) we know that there must be a point c between 0 and 2 such that $g'(c) = \dfrac{g(2)-g(0)}{2-0}$, i.e. that the slope of the tangent line to the graph of g will equal 3.

b)



All you need to know about the graph of $f(x) = e^{-x^2}$ is that it decreases from 0 to 1.

$MAX = f(0) = 1 \qquad MIN = f(1) = \frac{1}{e}$

$(MIN)\cdot 1 \leq AREA \leq (MAX)\cdot 1$

$\Rightarrow \frac{1}{e} \leq Area \leq 1$

Only d) 0.75 satisfies this estimate.

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6) $a(t) = -t^2$ mi/hr²

$v(t) = -\dfrac{t^3}{3} + C$

a) $v(0) = C = v_0 \Rightarrow \boxed{v(t) = v_0 - \dfrac{t^3}{3}}$

$s(t) = v_0 t - \dfrac{t^4}{12} + K \qquad s(0) = K = 0$

$\therefore \boxed{s(t) = v_0 t - \dfrac{t^4}{12}}$ b)

At MAXIMUM DISPLACEMENT, (call time $= T$) we'll have $v = 0$ (MAX) and $s = 4$ (given)

So $v_0 - \dfrac{T^3}{3} = 0 \Rightarrow T = \sqrt[3]{3v_0}$ or $v_0 = \dfrac{T^3}{3}$

Algebra is a little easier to find T first $\Rightarrow 4 = \dfrac{T^3}{3}\cdot T - \dfrac{T^4}{12} = \dfrac{3T^4}{12}$

So $T^4 = 16 \Rightarrow T = 2$.

This gives $\boxed{v_0 = \frac{8}{3} = 2\frac{2}{3} \text{ mi/hr}}$

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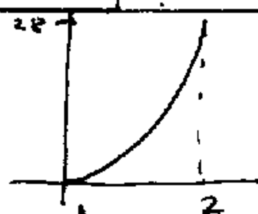
7) $f(x) = 3x^2 + 2x^3 \geq 0$

$f'(x) = 6x + 6x^2 \geq 0$

$f''(x) = 6 + 12x \geq 0$

for $x \in [0, 2]$ $\Rightarrow$



x	$f(x)$
0	0
$\frac{1}{2}$	$\frac{3}{4} + \frac{1}{4} = 1$
1	$3 + 2 = 5$
$\frac{3}{2}$	$\frac{27}{4} + \frac{27}{4} = \frac{27}{2}$
2	$12 + 16 = 28$

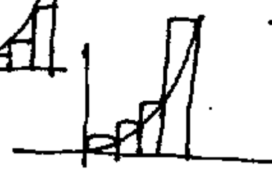
$n = 4$

a) $L = \frac{1}{2}\left(0 + 1 + 5 + \frac{27}{2}\right) = \frac{39}{4} = 9\frac{3}{4}$

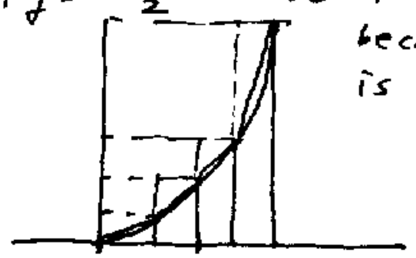
underestimate

b) $R = \frac{1}{2}\left(1 + 5 + \frac{27}{2} + 28\right) = \frac{95}{4} = 23\frac{3}{4}$

overestimate



c) $Avg = \dfrac{L+R}{2} = 16\frac{3}{4}$ overestimate because graph is concave up.



d) $\displaystyle\int_0^2 (3x^2 + 2x^3)\,dx$

$= \left[x^3 + \dfrac{x^4}{2}\right]_{x=0}^{x=2}$

$= (8+8) - (0) = \boxed{16}$