Name: SOLUTIONS

Math 1a Midterm Examination I—Wednesday, October 26, 2005

Please circle your section:

Bret Benesh	Tatyana Chmutova	Maksym Fedorchuk	Thomas Judson
MWF 9–10	10-11 MWF	10–11 MWF	11–12 MWF

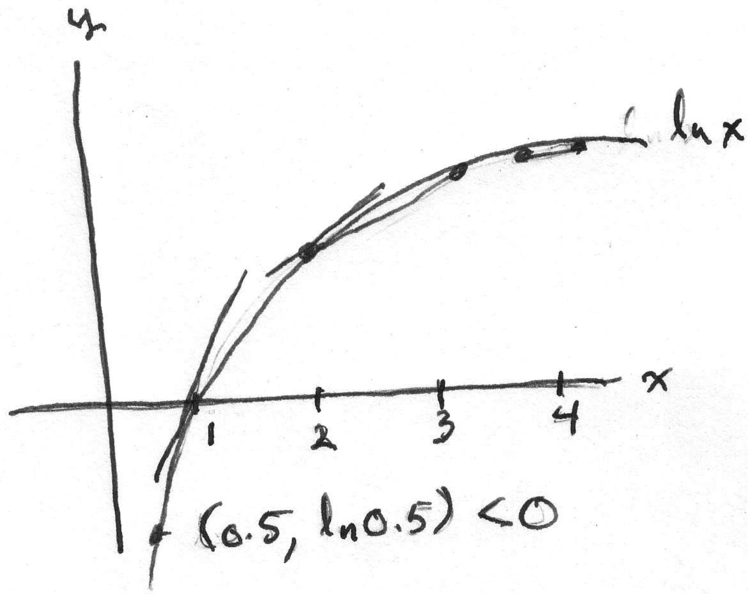
Tatyana Chmutova	Robin Gottlieb	Robin Gottlieb
MWF 12–1	TTh 10–11:30	TTh 11:30–1

Problem Number	Possible Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

Directions—Please Read Carefully! You have two hours to take this midterm. Make sure to use correct mathematical notation. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You do not have to answer the problems in any particular order, so move to another problem if you find you are stuck or spending too much time on a single problem. To receive full credit on a problem, you will need to justify your answers carefully—unsubstantiated answers will receive little or no credit (except if the directions for that question specifically say no justification is necessary, such as in a True/False section). Please be sure to write neatly—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers. *Calculators are not allowed.* **Good Luck!!!**

1. (10 points) Let $f(x) = \ln x$. By interpreting some of the following expressions as slopes of secant and tangent lines, list the following expressions in *ascending order* (from smallest number to largest).

   (a) $f'(1)$
   (b) $f'(2)$
   (c) $\ln 3 - \ln 2$
   (d) 0
   (e) $\ln 0.5$
   (f) $\dfrac{\ln 4 - \ln 3.5}{0.5}$



$\ln(0.5) < 0$

$< \dfrac{\ln 4 - \ln 3.5}{0.5}$ is the slope of a secant line through the points $(3.5, \ln 3.5)$ and $(4, \ln 4)$

$< \ln 3 - \ln 2$ is the slope of a secant line through the points $(2, \ln 2)$ and $(3, \ln 3)$

$< f'(2)$ is the slope of a tangent line at $(2, \ln 2)$

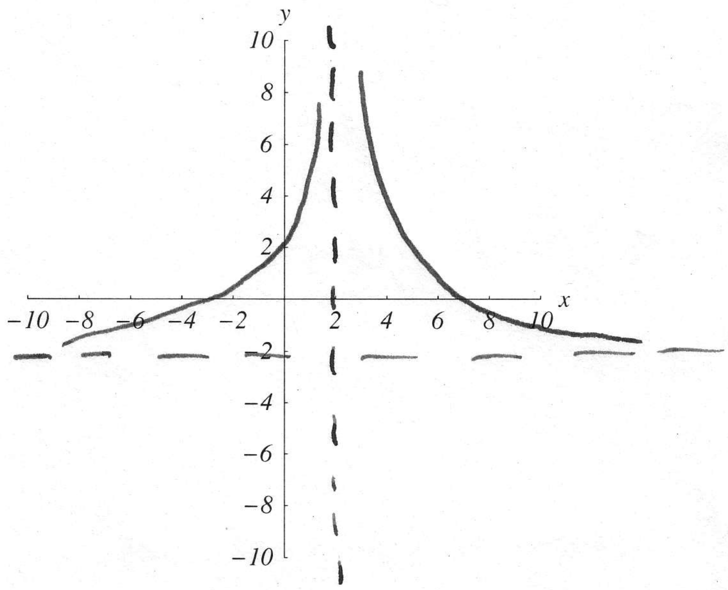
$< f'(1)$ is the slope of a tangent line at $(1, f(1))$

$$\ln(0.5) < 0 < \frac{\ln 4 - \ln 3.5}{0.5} < \ln 3 - \ln 2 < f'(2) < f'(1)$$

2. (10 points) A function f has *all* of the following characteristics simultaneously.

- The domain is all real numbers except $x = 2$ and whose range is $(-2, +\infty)$.
- $\lim_{x\to 2} f(x) = +\infty$
- $\lim_{x\to +\infty} f(x) = -2$ and $\lim_{x\to -\infty} f(x) = -2$
- f is continuous on $(-\infty, 2)$ and $(2, +\infty)$
- f is increasing on $(-\infty, 2)$ and decreasing on $(2, +\infty)$

(a) Sketch a possible graph of f.



(b) Give a possible formula for $f(x)$.

$$f(x) = \frac{1}{(x-2)^2} - 2$$

other solutions possible

(c) Which of the following statements accurately reflects the number of roots of any such function $f(x)$? (A value a is a root of $f(x)$ if $f(a) = 0$).

i. There are no roots

ii. There are exactly two roots

iii. There are at least two roots but there could be more roots

iv. There are no more than two roots, but there could be fewer roots

v. Not enough information to make one of the statements above.

3. (10 points)

(a) Write the limit definition of the derivative of a function f at a point $(c, f(c))$. (You should express the derivative as the limit of a difference quotient.)

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

or

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

(b) Using the limit definition of the derivative, find the derivative, $f'(x)$ of

$$f(x) = \frac{2x}{x+3}.$$

You will receive little credit for an answer obtained without explicit use of the limit of a difference quotient. Please show all your work.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

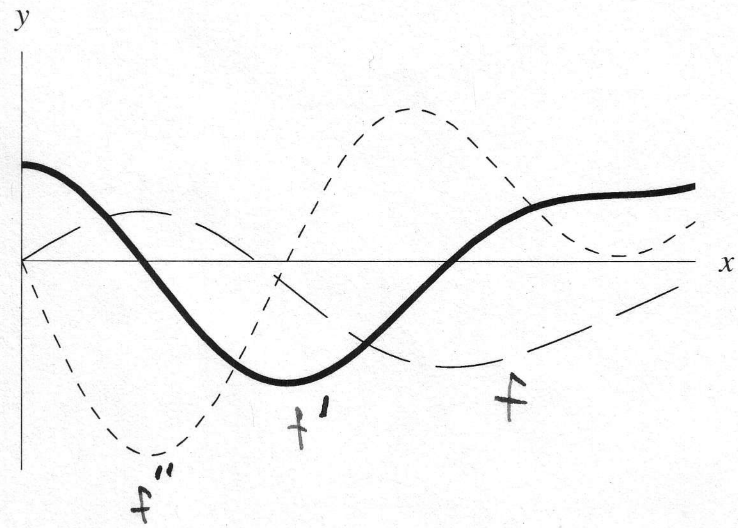
$$= \lim_{h \to 0} \frac{\frac{2(x+h)}{(x+h)+3} - \frac{2x}{x+3}}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h)(x+3) - 2x(x+h+3)}{h(x+h+3)(x+3)}$$

$$= \lim_{h \to 0} \frac{2x^2 + 2xh + 6x + 6h - 2x^2 - 2xh - 6x}{h(x+h+3)(x+3)}$$

$$= \lim_{h \to 0} \frac{6h}{h(x+h+3)(x+3)} = \frac{6}{(x+3)^2}$$

4. (10 points) The graph of function f is given below, as well as the graphs of the function's first and second derivatives, f' and f'', respectively. Indicate which graph belongs to which function.

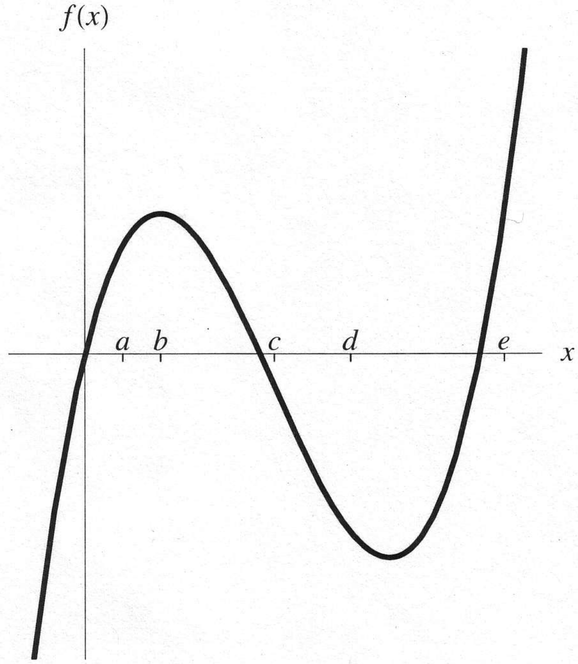


f'' must be "- - - - -" since its derivative function must intersect the x-axis at all maxima and minima. Neither "———" nor "— — —" fulfill this condition

For the same reason "- - - -" is the derivative function for "———" and so f is "— — —"

5. (10 points) The graph of f is shown below. Match the derivatives in the following table with the points a, b, c, d, and e on the graph below.

x	b	a	e	d	c
$f'(x)$	0	1.75	6.75	-1.25	-2.25



6. (10 points) Let $g(v)$ be the fuel efficiency, in miles per gallon, of a car going v miles per hour.

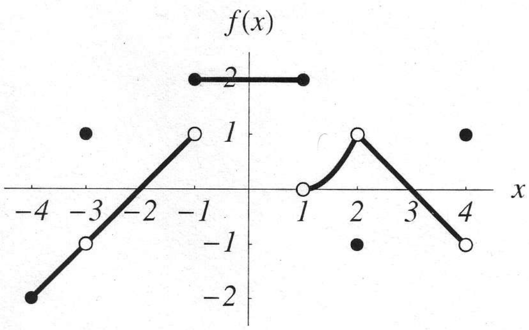
(a) What are the units of $g'(90)$?

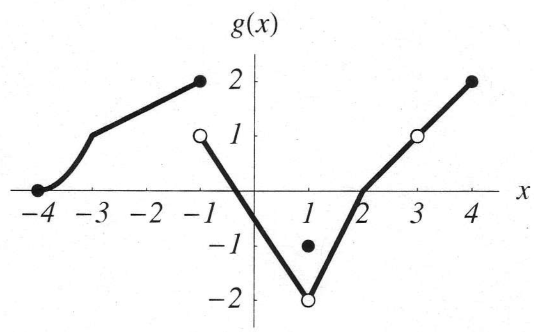
mpg/mph

(b) What is the practical meaning of the statement $g'(55) = -0.54$? Give a *brief* one or two sentence explanation.

At 55 mph the fuel efficiency of the car is going down at a rate of 0.54 mpg/mph

7. (10 points) Given the graphs of f and g below, evaluate each of the following limits. If the limit does not exist, say why.





(a) $\lim_{x\to 2} f(x)$

1

(b) $\lim_{x\to -1^+} g(x)$

1

(c) $\lim_{x\to -1^-} g(x)$

2

(d) $\lim_{x\to -1} f(x)+g(x)$

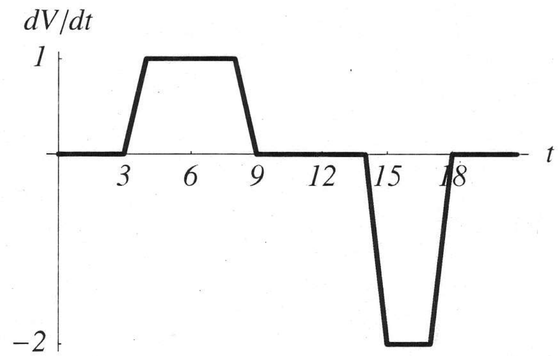
$$\lim_{x\to -1^+} f(x)+g(x) = 3 = \lim_{x\to -1^-} f(x)+g(x)$$

(e) $\lim_{x\to 3} \dfrac{f(x)}{x-3}$ $= \lim_{x\to 3} \dfrac{f(x)-f(3)}{x-3} = f'(3) = -1$

(f) $\lim_{h\to 0} \dfrac{g(-2+h)-3/2}{h}$ $= \lim_{h\to 0} \dfrac{g(-2+h)-g(-2)}{h}$

$$= g'(-2) = \frac{1}{2}$$

8. (10 points) A child inflates a balloon, admires it for a while and then lets the air out of the balloon at a constant rate. If $V(t)$ gives the volume of the balloon at time t, the figure below shows the graph of dV/dt as a function of t. At what time does the child:



(a) Begin to inflate the balloon?

$t = 3$

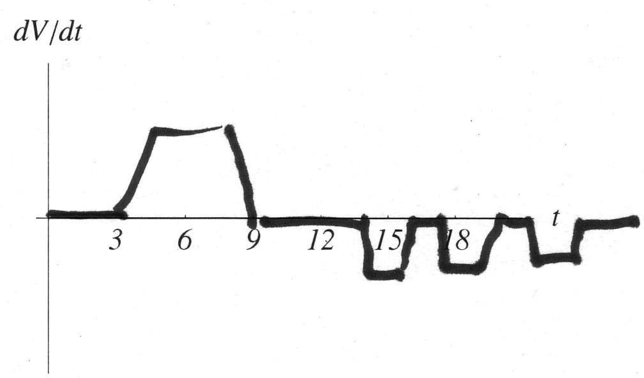
(b) Finishing inflating the balloon?

$t = 9$

(c) Begin to let the air out of the balloon?

$t = 14$

(d) What would the graph of $V'(t)$ look like if the child had alternated between pinching and releasing the open end of the balloon, instead of letting the air out at a constant rate?



9. (10 points)

(a) Evaluate

$$\lim_{x\to 0} x\cos\left(\frac{\pi}{x}\right).$$

$-1 \le \cos\left(\frac{\pi}{x}\right) \le 1 \Rightarrow -x \le x\cos\left(\frac{\pi}{x}\right) \le x$

$\Rightarrow$

$0 = \lim_{x\to 0} -x \le \lim_{x\to 0} x\cos\left(\frac{\pi}{x}\right) \le \lim_{x\to 0} x = 0$

By the Squeeze Theorem

$$\lim_{x\to 0} x\cos\left(\frac{\pi}{x}\right) = 0$$

(b) Let $f(x)$ be the piecewise defined function

$$f(x) = \begin{cases} x\cos(\pi/x) & \text{if } x \neq 0 \\ a & \text{if } x = 0 \end{cases}$$

Find a numerical value of a that will make $f(x)$ a continuous function.

By part (a), $a = 0$

10. (10 points) Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 0; \\ ax + b & \text{if } x \geq 0. \end{cases}$$

Find all the values of a and b such that

(a) $f(x)$ is continuous;

For $f(x)$ to be continuous at $x = 0$

$0^2 + 1 = a \cdot 0 + b$ or $b = 1$ and

a can be any real number

(b) $f(x)$ is differentiable.

$$f'(x) = \begin{cases} 2x, & \text{if } x < 0 \\ a, & \text{if } x \geq 0 \end{cases}$$

$\Rightarrow$ $a = 0$ for $f'(x)$ to exist at

$x = 0$.