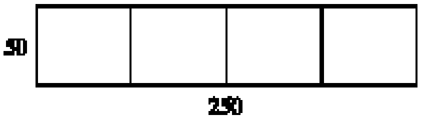
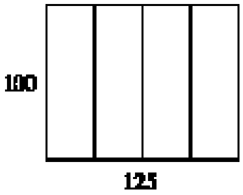
4. Let $x > 0$ and let $f(x) = x + 1/x$. We wish to minimize $f(x)$. Now $f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{1}{x^2}(x^2 - 1) = \dfrac{1}{x^2}(x+1)(x-1)$, so the only critical number in $(0, \infty)$ is 1.

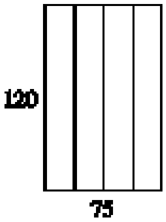
$f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$, so f has an absolute minimum at $x = 1$, and $f(1) = 2$.

Or: $f''(x) = 2/x^3 > 0$ for all $x > 0$, so f is concave upward everywhere and the critical point $(1, 2)$ must correspond to a local minimum for f.

7. (a)



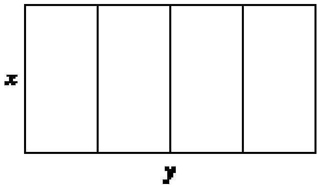




The areas of the three figures are 12,500, 12,500, and 9000 ft^2. There appears to be a maximum area of at least 12,500 ft^2.

(b) Let x denote the length of each of two sides and three dividers.

Let y denote the length of the other two sides.



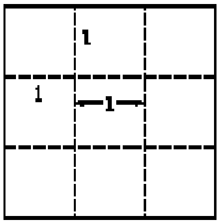
(c) Area $A = \text{length} \times \text{width} = y \cdot x$

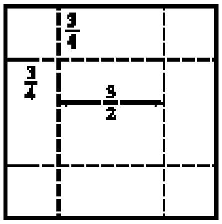
(d) Length of fencing $= 750 \quad \Rightarrow \quad 5x + 2y = 750$

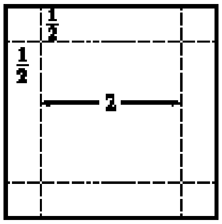
(e) $5x + 2y = 750 \quad \Rightarrow \quad y = 375 - \frac{5}{2}x \quad \Rightarrow \quad A(x) = \left(375 - \frac{5}{2}x\right)x = 375x - \frac{5}{2}x^2$

(f) $A'(x) = 375 - 5x = 0 \quad \Rightarrow \quad x = 75$. Since $A''(x) = -5 < 0$ there is an absolute maximum when $x = 75$. Then $y = \frac{375}{2} = 187.5$. The largest area is $75\left(\frac{375}{2}\right) = 14{,}062.5$ ft^2. These values of x and y are between the values in the first and second figures in part (a). Our original estimate was low.

8. (a)

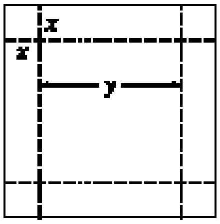






The volumes of the resulting boxes are 1, 1.6875, and 2 ft^3. There appears to be a maximum volume of at least 2 ft^3.

(b) Let x denote the length of the side of the square being cut out. Let y denote the length of the base.



(c) Volume V = length × width × height $\Rightarrow$ $V = y \cdot y \cdot x = xy^2$

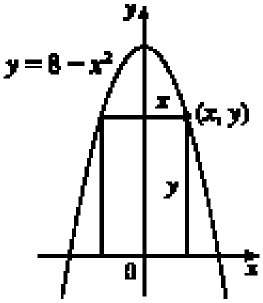
(d) Length of cardboard $= 3$ $\Rightarrow$ $x + y + x = 3$ $\Rightarrow$ $y + 2x = 3$

(e) $y + 2x = 3$ $\Rightarrow$ $y = 3 - 2x$ $\Rightarrow$ $V(x) = x(3-2x)^2$

(f) $V(x) = x(3-2x)^2$ $\Rightarrow$
$V'(x) = x \cdot 2(3-2x)(-2) + (3-2x)^2 \cdot 1 = (3-2x)[-4x + (3-2x)] = (3-2x)(-6x+3)$,
so the critical numbers are $x = \frac{3}{2}$ and $x = \frac{1}{2}$. Now $0 \le x \le \frac{3}{2}$ and $V(0) = V\left(\frac{3}{2}\right) = 0$, so the maximum is
$V\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)(2)^2 = 2$ ft^3, which is the value found from our third figure in part (a).

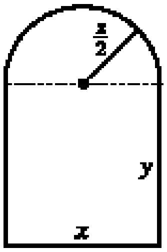
9. Let b be the length of the base of the box and h the height. The surface area is $1200 = b^2 + 4hb$ $\Rightarrow$
$h = \left(1200 - b^2\right)/(4b)$. The volume is $V = b^2h = b^2\left(1200 - b^2\right)/4b = 300b - b^3/4$ $\Rightarrow$ $V'(b) = 300 - \frac{3}{4}b^2$.
$V'(b) = 0$ $\Rightarrow$ $300 = \frac{3}{4}b^2$ $\Rightarrow$ $b^2 = 400$ $\Rightarrow$ $b = \sqrt{400} = 20$. Since $V'(b) > 0$ for $0 < b < 20$ and $V'(b) < 0$ for $b > 20$, there is an absolute maximum when $b = 20$ by the First Derivative Test for Absolute Extreme Values (see page 280).
If $b = 20$, then $h = \left(1200 - 20^2\right)/(4 \cdot 20) = 10$, so the largest possible volume is $b^2h = (20)^2(10) = 4000$ cm^3.

16.



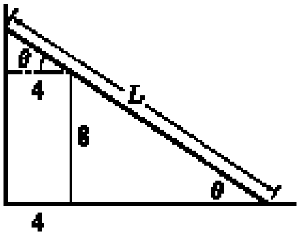
The rectangle has area $A(x) = 2xy = 2x\left(8 - x^2\right) = 16x - 2x^3$, where $0 \le x \le 2\sqrt{2}$. Now $A'(x) = 16 - 6x^2 = 0$ $\Rightarrow$ $x = 2\sqrt{\frac{2}{3}}$. Since $A(0) = A\left(2\sqrt{2}\right) = 0$, there is a maximum when $x = 2\sqrt{\frac{2}{3}}$. Then $y = \frac{16}{3}$, so the rectangle has dimensions $4\sqrt{\frac{2}{3}}$ and $\frac{16}{3}$.

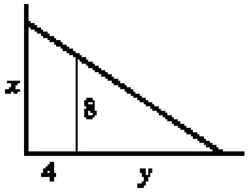
19.



Perimeter $= 30 \Rightarrow 2y + x + \pi\left(\frac{x}{2}\right) = 30 \Rightarrow$
$y = \frac{1}{2}\left(30 - x - \frac{\pi x}{2}\right) = 15 - \frac{x}{2} - \frac{\pi x}{4}$. The area is the area of the rectangle plus the area of the semicircle, or $xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$, so
$A(x) = x\left(15 - \frac{x}{2} - \frac{\pi x}{4}\right) + \frac{1}{8}\pi x^2 = 15x - \frac{1}{2}x^2 - \frac{\pi}{8}x^2$.
$A'(x) = 15 - \left(1 + \frac{\pi}{4}\right)x = 0 \quad\Rightarrow\quad x = \frac{15}{1 + \pi/4} = \frac{60}{4 + \pi}$. $A''(x) = -\left(1 + \frac{\pi}{4}\right) < 0$, so this gives a maximum. The dimensions are $x = \frac{60}{4+\pi}$ ft and $y = 15 - \frac{30}{4+\pi} - \frac{15\pi}{4+\pi} = \frac{60 + 15\pi - 30 - 15\pi}{4+\pi} = \frac{30}{4+\pi}$ ft, so the height of the rectangle is half the base.

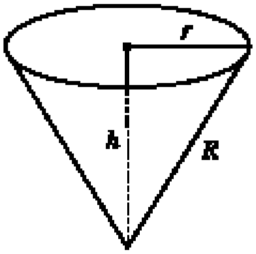
22.





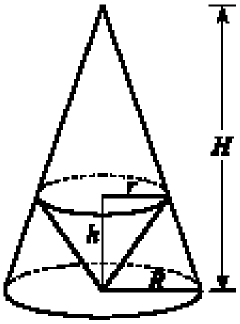
$L = 8\csc\theta + 4\sec\theta, 0 < \theta < \frac{\pi}{2}$,
$\frac{dL}{d\theta} = -8\csc\theta\,\cot\theta + 4\sec\theta\,\tan\theta = 0$ when
$\sec\theta\,\tan\theta = 2\csc\theta\,\cot\theta \quad\Leftrightarrow\quad \tan^3\theta = 2 \quad\Leftrightarrow\quad \tan\theta = \sqrt[3]{2} \quad\Leftrightarrow$
$\theta = \tan^{-1}\sqrt[3]{2}$.
$dL/d\theta < 0$ when $0 < \theta < \tan^{-1}\sqrt[3]{2}$, $dL/d\theta > 0$ when $\tan^{-1}\sqrt[3]{2} < \theta < \frac{\pi}{2}$, so L has an absolute minimum when $\theta = \tan^{-1}\sqrt[3]{2}$, and the shortest ladder has length
$L = 8\frac{\sqrt{1 + 2^{2/3}}}{2^{1/3}} + 4\sqrt{1 + 2^{2/3}} \approx 16.65$ ft.
Another method: Minimize $L^2 = x^2 + (4+y)^2$, where $\frac{x}{4+y} = \frac{8}{y}$.

23.



$h^2 + r^2 = R^2 \quad\Rightarrow\quad V = \frac{\pi}{3}r^2h = \frac{\pi}{3}(R^2 - h^2)h = \frac{\pi}{3}(R^2h - h^3)$.
$V'(h) = \frac{\pi}{3}(R^2 - 3h^2) = 0$ when $h = \frac{1}{\sqrt{3}}R$. This gives an absolute maximum, since $V'(h) > 0$ for $0 < h < \frac{1}{\sqrt{3}}R$ and $V'(h) < 0$ for $h > \frac{1}{\sqrt{3}}R$. The maximum volume is
$V\left(\frac{1}{\sqrt{3}}R\right) = \frac{\pi}{3}\left(\frac{1}{\sqrt{3}}R^3 - \frac{1}{3\sqrt{3}}R^3\right) = \frac{2}{9\sqrt{3}}\pi R^3$.

25.



By similar triangles, $\dfrac{H}{R} = \dfrac{H-h}{r}$ **(1)**. The volume of the inner cone is $V = \frac{1}{3}\pi r^2 h$, so we'll solve **(1)** for h. $\dfrac{Hr}{R} = H - h \quad \Rightarrow$

$$h = H - \frac{Hr}{R} = \frac{HR - Hr}{R} = \frac{H}{R}(R - r) \quad \textbf{(2)}.$$

Thus, $V(r) = \dfrac{\pi}{3}r^2 \cdot \dfrac{H}{R}(R - r) = \dfrac{\pi H}{3R}(Rr^2 - r^3) \quad \Rightarrow$

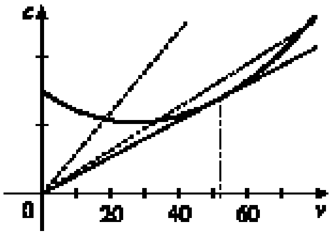
$$V'(r) = \frac{\pi H}{3R}(2Rr - 3r^2) = \frac{\pi H}{3R}r(2R - 3r).$$

$V'(r) = 0 \quad \Rightarrow \quad r = 0$ or $2R = 3r \quad \Rightarrow \quad r = \frac{2}{3}R$ and from **(2)**, $h = \dfrac{H}{R}\left(R - \frac{2}{3}R\right) = \dfrac{H}{R}\left(\frac{1}{3}R\right) = \frac{1}{3}H$.

$V'(r)$ changes from positive to negative at $r = \frac{2}{3}R$, so the inner cone has a maximum volume of

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{3}R\right)^2\left(\frac{1}{3}H\right) = \frac{4}{27} \cdot \frac{1}{3}\pi R^2 H$, which is approximately 15% of the volume of the larger cone.

26. We note that since c is the consumption in gallons per hour, and v is the velocity in miles per hour, then $\dfrac{c}{v} = \dfrac{\text{gallons/hour}}{\text{miles/hour}} = \dfrac{\text{gallons}}{\text{mile}}$ gives us the consumption in gallons per mile, that is, the quantity G. To find the minimum, we calculate $\dfrac{dG}{dv} = \dfrac{d}{dv}\left(\dfrac{c}{v}\right) = \dfrac{v\dfrac{dc}{dv} - c\dfrac{dv}{dv}}{v^2} = \dfrac{v\dfrac{dc}{dv} - c}{v^2}$.



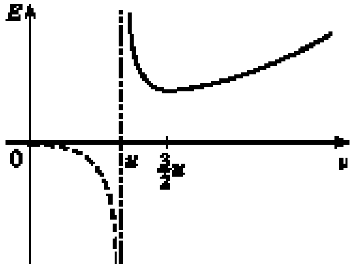
This is 0 when $v\dfrac{dc}{dv} - c = 0 \quad \Leftrightarrow \quad \dfrac{dc}{dv} = \dfrac{c}{v}$. This implies that the tangent line of $c(v)$ passes through the origin, and this occurs when $v \approx 53$ mi/h. Note that the slope of the secant line through the origin and a point $(v, c(v))$ on the graph is equal to $G(v)$, and it is intuitively clear that G is minimized in the case where the secant is in fact a tangent.

28. (a) $E(v) = \dfrac{aLv^3}{v - u} \quad \Rightarrow \quad E'(v) = aL\dfrac{(v-u)3v^2 - v^3}{(v-u)^2} = 0$ when

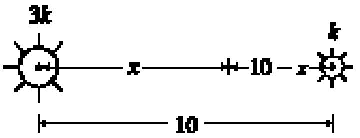
$2v^3 = 3uv^2 \quad \Rightarrow \quad 2v = 3u \quad \Rightarrow \quad v = \frac{3}{2}u$.

The First Derivative Test shows that this value of v gives the minimum value of E.

(b)



31.



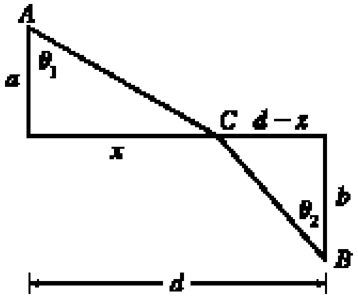
The total illumination is $I(x) = \dfrac{3k}{x^2} + \dfrac{k}{(10-x)^2}$, $0 < x < 10$. Then

$$I'(x) = \frac{-6k}{x^3} + \frac{2k}{(10-x)^3} = 0 \quad \Rightarrow \quad 6k(10-x)^3 = 2kx^3 \quad \Rightarrow$$

$$3(10-x)^3 = x^3 \quad \Rightarrow \quad \sqrt[3]{3}\,(10-x) = x \quad \Rightarrow \quad 10\sqrt[3]{3} - \sqrt[3]{3}\,x = x \quad \Rightarrow \quad 10\sqrt[3]{3} = x + \sqrt[3]{3}\,x \quad \Rightarrow$$

$10\sqrt[3]{3} = \left(1+\sqrt[3]{3}\right)x \quad \Rightarrow \quad x = \dfrac{10\sqrt[3]{3}}{1+\sqrt[3]{3}} \approx 5.9$ ft. This gives a minimum since $I''(x) > 0$ for $0 < x < 10$.

37.



The total time is

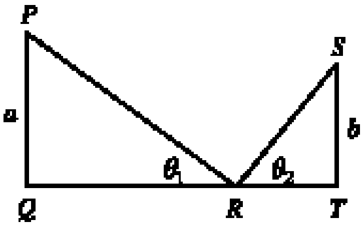
$$\begin{aligned} T(x) &= (\text{time from } A \text{ to } C) + (\text{time from } C \text{ to } B) \\ &= \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(d-x)^2}}{v_2}, \quad 0 < x < d \end{aligned}$$

$$T'(x) = \frac{x}{v_1\sqrt{a^2+x^2}} - \frac{d-x}{v_2\sqrt{b^2+(d-x)^2}} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

The minimum occurs when $T'(x) = 0 \quad \Rightarrow \quad \dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$.

[*Note:* $T''(x) > 0$]

38.



If $d = |QT|$, we minimize $f(\theta_1) = |PR| + |RS| = a\csc\theta_1 + b\csc\theta_2$.

Differentiating with respect to θ_1, and setting $\dfrac{df}{d\theta_1}$ equal to 0, we get

$$\frac{df}{d\theta_1} = 0 = -a\csc\theta_1\cot\theta_1 - b\csc\theta_2\cot\theta_2\,\frac{d\theta_2}{d\theta_1}.$$

So we need to find an expression for $\dfrac{d\theta_2}{d\theta_1}$. We can do this by observing that $|QT| = \text{constant} = a\cot\theta_1 + b\cot\theta_2$.

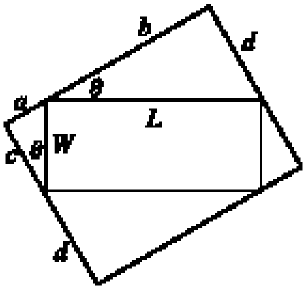
Differentiating this equation implicitly with respect to θ_1, we get $-a\csc^2\theta_1 - b\csc^2\theta_2\,\dfrac{d\theta_2}{d\theta_1} = 0 \quad \Rightarrow$

$\dfrac{d\theta_2}{d\theta_1} = -\dfrac{a\csc^2\theta_1}{b\csc^2\theta_2}$. We substitute this into the expression for $\dfrac{df}{d\theta_1}$ to get

$$-a\csc\theta_1\cot\theta_1 - b\csc\theta_2\cot\theta_2\left(-\frac{a\csc^2\theta_1}{b\csc^2\theta_2}\right) = 0 \quad \Leftrightarrow \quad -a\csc\theta_1\cot\theta_1 + a\frac{\csc^2\theta_1\cot\theta_2}{\csc\theta_2} = 0 \quad \Leftrightarrow$$

$\cot\theta_1\csc\theta_2 = \csc\theta_1\cot\theta_2 \quad \Leftrightarrow \quad \dfrac{\cot\theta_1}{\csc\theta_1} = \dfrac{\cot\theta_2}{\csc\theta_2} \quad \Leftrightarrow \quad \cos\theta_1 = \cos\theta_2$. Since θ_1 and θ_2 are both acute, we have $\theta_1 = \theta_2$.

41.



In the small triangle with sides a and c and hypotenuse W, $\sin\theta = \dfrac{a}{W}$ and $\cos\theta = \dfrac{c}{W}$. In the triangle with sides b and d and hypotenuse L, $\sin\theta = \dfrac{d}{L}$ and $\cos\theta = \dfrac{b}{L}$. Thus, $a = W\sin\theta$, $c = W\cos\theta$, $d = L\sin\theta$, and $b = L\cos\theta$, so the area of the circumscribed rectangle is

$$\begin{aligned}A(\theta) &= (a+b)(c+d) = (W\sin\theta + L\cos\theta)(W\cos\theta + L\sin\theta)\\ &= W^2\sin\theta\,\cos\theta + WL\sin^2\theta + LW\cos^2\theta + L^2\sin\theta\,\cos\theta\\ &= LW\sin^2\theta + LW\cos^2\theta + \left(L^2+W^2\right)\sin\theta\,\cos\theta\\ &= LW\left(\sin^2\theta + \cos^2\theta\right) + \left(L^2+W^2\right)\cdot\tfrac{1}{2}\cdot 2\sin\theta\,\cos\theta\\ &= LW + \tfrac{1}{2}\left(L^2+W^2\right)\sin 2\theta, \quad 0 \le \theta \le \tfrac{\pi}{2}\end{aligned}$$

This expression shows, without calculus, that the maximum value of $A(\theta)$ occurs when $\sin 2\theta = 1 \quad\Leftrightarrow\quad 2\theta = \frac{\pi}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{4}$. So the maximum area is

$A\left(\frac{\pi}{4}\right) = LW + \frac{1}{2}\left(L^2+W^2\right) = \frac{1}{2}\left(L^2 + 2LW + W^2\right) = \frac{1}{2}(L+W)^2$.

46. (a) Let D be the point such that $a = |AD|$. From the figure, $\sin\theta = \dfrac{b}{|BC|} \quad\Rightarrow\quad |BC| = b\csc\theta$ and

$\cos\theta = \dfrac{|BD|}{|BC|} = \dfrac{a - |AB|}{|BC|} \quad\Rightarrow\quad |BC| = (a - |AB|)\sec\theta$. Eliminating $|BC|$ gives

$(a - |AB|)\sec\theta = b\csc\theta \quad\Rightarrow\quad b\cot\theta = a - |AB| \quad\Rightarrow\quad |AB| = a - b\cot\theta$. The total resistance is

$R(\theta) = C\dfrac{|AB|}{r_1^4} + C\dfrac{|BC|}{r_2^4} = C\left(\dfrac{a - b\cot\theta}{r_1^4} + \dfrac{b\csc\theta}{r_2^4}\right)$.

(b) $R'(\theta) = C\left(\dfrac{b\csc^2\theta}{r_1^4} - \dfrac{b\csc\theta\,\cot\theta}{r_2^4}\right) = bC\csc\theta\left(\dfrac{\csc\theta}{r_1^4} - \dfrac{\cot\theta}{r_2^4}\right)$.

$R'(\theta) = 0 \quad\Leftrightarrow\quad \dfrac{\csc\theta}{r_1^4} = \dfrac{\cot\theta}{r_2^4} \quad\Leftrightarrow\quad \dfrac{r_2^4}{r_1^4} = \dfrac{\cot\theta}{\csc\theta} = \cos\theta$.

$R'(\theta) > 0 \quad\Leftrightarrow\quad \dfrac{\csc\theta}{r_1^4} > \dfrac{\cot\theta}{r_2^4} \quad\Rightarrow\quad \cos\theta < \dfrac{r_2^4}{r_1^4}$ and $R'(\theta) < 0$ when $\cos\theta > \dfrac{r_2^4}{r_1^4}$, so there is an absolute minimum when $\cos\theta = r_2^4/r_1^4$.

(c) When $r_2 = \frac{2}{3}r_1$, we have $\cos\theta = \left(\frac{2}{3}\right)^4$, so $\theta = \cos^{-1}\left(\frac{2}{3}\right)^4 \approx 79°$.