

1. From Snell's Law, we have $\sin \alpha = k \sin \beta \approx \frac{4}{3} \sin \beta \Leftrightarrow \beta \approx \arcsin\left(\frac{3}{4} \sin \alpha\right)$. We substitute this into $D(\alpha) = \pi + 2\alpha - 4\beta = \pi + 2\alpha - 4 \arcsin\left(\frac{3}{4} \sin \alpha\right)$, and then differentiate to find the minimum:

$$D'(\alpha) = 2 - 4 \left[1 - \left(\frac{3}{4} \sin \alpha\right)^2\right]^{-1/2} \left(\frac{3}{4} \cos \alpha\right) = 2 - \frac{3 \cos \alpha}{\sqrt{1 - \frac{9}{16} \sin^2 \alpha}}. \text{ This is 0 when } \frac{3 \cos \alpha}{\sqrt{1 - \frac{9}{16} \sin^2 \alpha}} = 2 \Leftrightarrow$$

$$\frac{9}{4} \cos^2 \alpha = 1 - \frac{9}{16} \sin^2 \alpha \Leftrightarrow \frac{9}{4} \cos^2 \alpha = 1 - \frac{9}{16} (1 - \cos^2 \alpha) \Leftrightarrow \frac{27}{16} \cos^2 \alpha = \frac{7}{16} \Leftrightarrow \cos \alpha = \sqrt{\frac{7}{27}} \Leftrightarrow \alpha = \arccos \sqrt{\frac{7}{27}} \approx 59.4^\circ, \text{ and so the local minimum is } D(59.4^\circ) \approx 2.4 \text{ radians} \approx 138^\circ.$$

To see that this is an absolute minimum, we check the endpoints, which in this case are $\alpha = 0$ and $\alpha = \frac{\pi}{2}$:

$$D(0) = \pi \text{ radians} = 180^\circ, \text{ and } D\left(\frac{\pi}{2}\right) \approx 166^\circ.$$

Another method: We first calculate $\frac{d\beta}{d\alpha}$: $\sin \alpha = \frac{4}{3} \sin \beta \Leftrightarrow \cos \alpha = \frac{4}{3} \cos \beta \frac{d\beta}{d\alpha} \Leftrightarrow \frac{d\beta}{d\alpha} = \frac{3 \cos \alpha}{4 \cos \beta}$, so since

$D'(\alpha) = 2 - 4 \frac{d\beta}{d\alpha} = 0 \Leftrightarrow \frac{d\beta}{d\alpha} = \frac{1}{2}$, the minimum occurs when $3 \cos \alpha = 2 \cos \beta$. Now we square both sides and substitute $\sin \alpha = \frac{4}{3} \sin \beta$, leading to the same result.

2. If we repeat Problem 1 with k in place of $\frac{4}{3}$, we get $D(\alpha) = \pi + 2\alpha - 4 \arcsin\left(\frac{1}{k} \sin \alpha\right) \Rightarrow$

$$D'(\alpha) = 2 - \frac{4 \cos \alpha}{k \sqrt{1 - \left[\frac{\sin \alpha}{k}\right]^2}}, \text{ which is 0 when } \frac{2 \cos \alpha}{k} = \sqrt{1 - \left(\frac{\sin \alpha}{k}\right)^2} \Leftrightarrow \left(\frac{2 \cos \alpha}{k}\right)^2 = 1 - \left(\frac{\sin \alpha}{k}\right)^2$$

$\Leftrightarrow 4 \cos^2 \alpha = k^2 - \sin^2 \alpha \Leftrightarrow 3 \cos^2 \alpha = k^2 - 1 \Leftrightarrow \alpha = \arccos \sqrt{\frac{k^2 - 1}{3}}$. So for $k \approx 1.3318$ (red light) the minimum occurs at $\alpha_1 \approx 1.038$ radians, and so the rainbow angle is about $\pi - D(\alpha_1) \approx 42.3^\circ$. For $k \approx 1.3435$ (violet light) the minimum occurs at $\alpha_2 \approx 1.026$ radians, and so the rainbow angle is about $\pi - D(\alpha_2) \approx 40.6^\circ$.

Another method: As in Problem 1, we can instead find $D'(\alpha)$ in terms of $\frac{d\beta}{d\alpha}$, and then substitute $\frac{d\beta}{d\alpha} = \frac{\cos \alpha}{k \cos \beta}$.

3. At each reflection or refraction, the light is bent in a counterclockwise direction: the bend at A is $\alpha - \beta$, the bend at B is $\pi - 2\beta$, the bend at C is again $\pi - 2\beta$, and the bend at D is $\alpha - \beta$. So the total bend is

$D(\alpha) = 2(\alpha - \beta) + 2(\pi - 2\beta) = 2\alpha - 6\beta + 2\pi$, as required. We substitute $\beta = \arcsin\left(\frac{\sin \alpha}{k}\right)$ and differentiate, to get

$$D'(\alpha) = 2 - \frac{6 \cos \alpha}{k \sqrt{1 - [(\sin \alpha)/k]^2}}, \text{ which is 0 when } \frac{3 \cos \alpha}{k} = \sqrt{1 - \left(\frac{\sin \alpha}{k}\right)^2} \Leftrightarrow 9 \cos^2 \alpha = k^2 - \sin^2 \alpha \Leftrightarrow$$

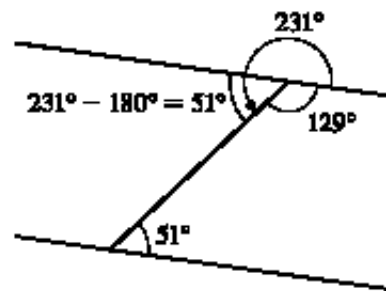
$8 \cos^2 \alpha = k^2 - 1 \Leftrightarrow \cos \alpha = \sqrt{\frac{1}{8}(k^2 - 1)}$. If $k = \frac{4}{3}$, then the minimum occurs at

$$\alpha_1 = \arccos \sqrt{\frac{(4/3)^2 - 1}{8}} \approx 1.254 \text{ radians. Thus, the minimum}$$

counterclockwise rotation is $D(\alpha_1) \approx 231^\circ$, which is equivalent to a clockwise rotation of $360^\circ - 231^\circ = 129^\circ$ (see the figure). So the rainbow angle for the secondary rainbow is about $180^\circ - 129^\circ = 51^\circ$, as required.

In general, the rainbow angle for the secondary rainbow is

$$\pi - [2\pi - D(\alpha)] = D(\alpha) - \pi.$$



4. In the primary rainbow, the rainbow angle gets smaller as k gets larger, as we found in Problem 2, so the colors appear from top to bottom in order of increasing k . But in the secondary rainbow, the rainbow angle gets larger as k gets larger. To see this, we find the minimum deviations for red light and for violet light in the secondary rainbow. For $k \approx 1.3318$ (red light) the

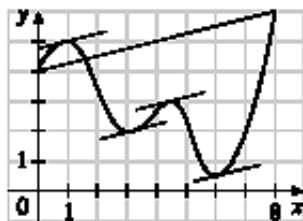
minimum occurs at $\alpha_1 \approx \arccos \sqrt{\frac{1.3318^2 - 1}{8}} \approx 1.255$ radians, and so the rainbow angle is $D(\alpha_1) - \pi \approx 50.6^\circ$. For

$k \approx 1.3435$ (violet light) the minimum occurs at $\alpha_2 \approx \arccos \sqrt{\frac{1.3435^2 - 1}{8}} \approx 1.248$ radians, and so the rainbow angle is

$D(\alpha_2) - \pi \approx 53.6^\circ$. Consequently, the rainbow angle is larger for colors with higher indices of refraction, and the colors appear from bottom to top in order of increasing k , the reverse of their order in the primary rainbow.

Note that our calculations above also explain why the secondary rainbow is more spread out than the primary rainbow: in the primary rainbow, the difference between rainbow angles for red and violet light is about 1.7° , whereas in the secondary rainbow it is about 3° .

1. $\frac{f(8) - f(0)}{8 - 0} = \frac{6 - 4}{8} = \frac{1}{4}$. The values of c which satisfy $f'(c) = \frac{1}{4}$ seem to be about $c = 0.8, 3.2, 4.4,$ and 6.1 .



2. (a) g is concave upward on $(-1, 2)$ and $(7, 8)$.
 (b) g is concave downward on $(2, 4)$ and $(4, 7)$.
 (c) The only point of inflection is $(2, 2)$. Note that 7 is not in the domain of this function.
3. (a) Use the Increasing/Decreasing (I/D) Test.
 (b) Use the Concavity Test.
 (c) At any value of x where the concavity changes, we have an inflection point at $(x, f(x))$.
8. (a) $f(x) = x^4 - 4x - 1 \Rightarrow f'(x) = 4x^3 - 4 = 4(x^3 - 1) = 4(x - 1)(x^2 + x + 1)$. So $f'(x) > 0 \Leftrightarrow x - 1 > 0$ [$4(x^2 + x + 1) > 0$] $\Leftrightarrow x > 1$. Thus, f is increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$.
 (b) f changes from decreasing to increasing at its only critical number, $x = 1$. Thus, $f(1) = -4$ is a local minimum value.
 (c) $f'(x) = 4x^3 - 4 \Rightarrow f''(x) = 12x^2$. $f''(x) > 0$ for all x except $x = 0$. Thus, f is concave upward on $(-\infty, 0)$ and $(0, \infty)$. Moreover, since f' is increasing on $(-\infty, \infty)$, f is concave upward on $(-\infty, \infty)$. There are no inflection points.

16. $f(x) = \frac{x}{x^2 + 4} \Rightarrow f'(x) = \frac{(x^2 + 4) \cdot 1 - x(2x)}{(x^2 + 4)^2} = \frac{4 - x^2}{(x^2 + 4)^2} = \frac{(2 + x)(2 - x)}{(x^2 + 4)^2}$.

First Derivative Test: $f'(x) > 0 \Rightarrow -2 < x < 2$ and $f'(x) < 0 \Rightarrow x > 2$ or $x < -2$. Since f' changes from positive to negative at $x = 2$, $f(2) = \frac{1}{4}$ is a local maximum value; and since f' changes from negative to positive at $x = -2$, $f(-2) = -\frac{1}{4}$ is a local minimum value.

Second Derivative Test:

$$\begin{aligned} f''(x) &= \frac{(x^2 + 4)^2(-2x) - (4 - x^2) \cdot 2(x^2 + 4)(2x)}{[(x^2 + 4)^2]^2} \\ &= \frac{-2x(x^2 + 4)[(x^2 + 4) + 2(4 - x^2)]}{(x^2 + 4)^4} = \frac{-2x(12 - x^2)}{(x^2 + 4)^3} \end{aligned}$$

$f'(x) = 0 \Leftrightarrow x = \pm 2$. $f''(-2) = \frac{1}{16} > 0 \Rightarrow f(-2) = -\frac{1}{4}$ is a local minimum value.

$f''(2) = -\frac{1}{16} < 0 \Rightarrow f(2) = \frac{1}{4}$ is a local maximum value.

Preference: Since calculating the second derivative is fairly difficult, the First Derivative Test is easier to use for this function.

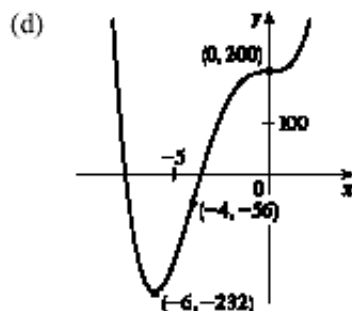
20. (a) $g(x) = 200 + 8x^3 + x^4 \Rightarrow g'(x) = 24x^2 + 4x^3 = 4x^2(6 + x) = 0$ when $x = -6$ and when $x = 0$. $g'(x) > 0 \Leftrightarrow x > -6$ ($x \neq 0$) and $g'(x) < 0 \Leftrightarrow x < -6$, so g is decreasing on $(-\infty, -6)$ and g is increasing on $(-6, \infty)$, with a horizontal tangent at $x = 0$.

- (b) $g(-6) = -232$ is a local minimum value.

There is no local maximum value.

- (c) $g''(x) = 48x + 12x^2 = 12x(4 + x) = 0$ when $x = -4$ and when $x = 0$.

$g''(x) > 0 \Leftrightarrow x < -4$ or $x > 0$ and $g''(x) < 0 \Leftrightarrow -4 < x < 0$, so g is CU on $(-\infty, -4)$ and $(0, \infty)$, and g is CD on $(-4, 0)$. Inflection points at $(-4, -56)$ and $(0, 200)$



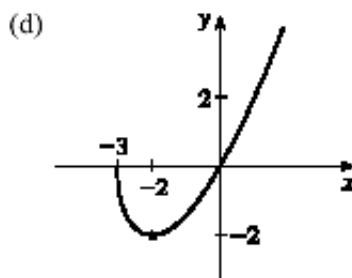
23. (a) $A(x) = x\sqrt{x+3} \Rightarrow A'(x) = x \cdot \frac{1}{2}(x+3)^{-1/2} + \sqrt{x+3} \cdot 1 = \frac{x}{2\sqrt{x+3}} + \sqrt{x+3} = \frac{x+2(x+3)}{2\sqrt{x+3}} = \frac{3x+6}{2\sqrt{x+3}}$.

The domain of A is $[-3, \infty)$. $A'(x) > 0$ for $x > -2$ and $A'(x) < 0$ for $-3 < x < -2$, so A is increasing on $(-2, \infty)$ and decreasing on $(-3, -2)$.

- (b) $A(-2) = -2$ is a local minimum value.

(c)
$$A''(x) = \frac{2\sqrt{x+3} \cdot 3 - (3x+6) \cdot \frac{1}{\sqrt{x+3}}}{(2\sqrt{x+3})^2}$$

$$= \frac{6(x+3) - (3x+6)}{4(x+3)^{3/2}} = \frac{3x+12}{4(x+3)^{3/2}} = \frac{3(x+4)}{4(x+3)^{3/2}}$$



$A''(x) > 0$ for all $x > -3$, so A is concave upward on $(-3, \infty)$. There is no inflection point.

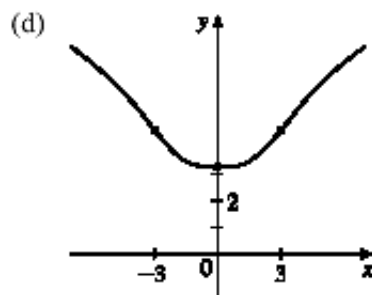
26. (a) $f(x) = \ln(x^4 + 27) \Rightarrow f'(x) = \frac{4x^3}{x^4 + 27}$. $f'(x) > 0$ if $x > 0$ and $f'(x) < 0$ if $x < 0$, so f is increasing on $(0, \infty)$ and f is decreasing on $(-\infty, 0)$.

(b) $f(0) = \ln 27 \approx 3.3$ is a local minimum value.

$$(c) f''(x) = \frac{(x^4 + 27)(12x^2) - 4x^3(4x^3)}{(x^4 + 27)^2} = \frac{4x^2[3(x^4 + 27) - 4x^4]}{(x^4 + 27)^2}$$

$$= \frac{4x^2(81 - x^4)}{(x^4 + 27)^2} = \frac{-4x^2(x^2 + 9)(x + 3)(x - 3)}{(x^4 + 27)^2}$$

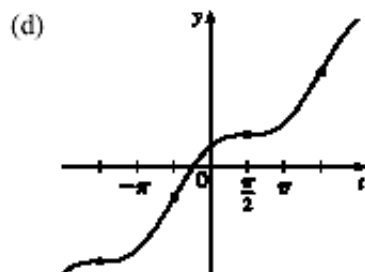
$f''(x) > 0$ if $-3 < x < 0$ and $0 < x < 3$, and $f''(x) < 0$ if $x < -3$ or $x > 3$. Thus, f is concave upward on $(-3, 0)$ and $(0, 3)$ [hence on $(-3, 3)$] and f is concave downward on $(-\infty, -3)$ and $(3, \infty)$. There are inflection points at $(\pm 3, \ln 108) \approx (\pm 3, 4.68)$.



28. (a) $f(t) = t + \cos t$, $-2\pi \leq t \leq 2\pi \Rightarrow f'(t) = 1 - \sin t \geq 0$ for all t and $f'(t) = 0$ when $\sin t = 1 \Leftrightarrow t = -\frac{3\pi}{2}$ or $\frac{\pi}{2}$, so f is increasing on $(-2\pi, 2\pi)$.

(b) No maximum or minimum

(c) $f''(t) = -\cos t > 0 \Leftrightarrow t \in (-\frac{3\pi}{2}, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2})$, so f is CU on these intervals and CD on $(-2\pi, -\frac{3\pi}{2})$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, and $(\frac{3\pi}{2}, 2\pi)$. Points of inflection at $(\pm\frac{3\pi}{2}, \pm\frac{3\pi}{2})$ and $(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$



30. $f(x) = \frac{x^2}{(x-2)^2}$ has domain $(-\infty, 2) \cup (2, \infty)$.

$$(a) \lim_{x \rightarrow \pm\infty} \frac{x^2}{x^2 - 4x + 4} = \lim_{x \rightarrow \pm\infty} \frac{x^2/x^2}{(x^2 - 4x + 4)/x^2} = \lim_{x \rightarrow \pm\infty} \frac{1}{1 - 4/x + 4/x^2} = \frac{1}{1 - 0 + 0} = 1,$$

so $y = 1$ is a HA. $\lim_{x \rightarrow 2^+} \frac{x^2}{(x-2)^2} = \infty$ since $x^2 \rightarrow 4$ and $(x-2)^2 \rightarrow 0^+$ as $x \rightarrow 2^+$, so $x = 2$ is a VA.

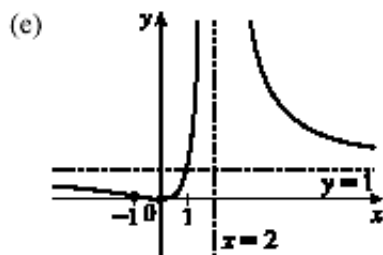
$$(b) f(x) = \frac{x^2}{(x-2)^2} \Rightarrow f'(x) = \frac{(x-2)^2(2x) - x^2 \cdot 2(x-2)}{[(x-2)^2]^2} = \frac{2x(x-2)[(x-2) - x]}{(x-2)^4} = \frac{-4x}{(x-2)^3}. f'(x) > 0 \text{ if}$$

$0 < x < 2$ and $f'(x) < 0$ if $x < 0$ or $x > 2$, so f is increasing on $(0, 2)$ and f is decreasing on $(-\infty, 0)$ and $(2, \infty)$.

(c) $f(0) = 0$ is a local minimum value.

$$(d) f''(x) = \frac{(x-2)^3(-4) - (-4x) \cdot 3(x-2)^2}{[(x-2)^3]^2} \\ = \frac{4(x-2)^2[-(x-2) + 3x]}{(x-2)^6} = \frac{8(x+1)}{(x-2)^4}$$

$f''(x) > 0$ if $x > -1$ ($x \neq 2$) and $f''(x) < 0$ if $x < -1$. Thus, f is CU on $(-1, 2)$ and $(2, \infty)$, and f is CD on $(-\infty, -1)$. There is an inflection point at $(-1, \frac{1}{9})$.



$$45. y = -\frac{W}{24EI}x^4 + \frac{WL}{12EI}x^3 - \frac{WL^2}{24EI}x^2 = -\frac{W}{24EI}x^2(x^2 - 2Lx + L^2) \\ = \frac{-W}{24EI}x^2(x-L)^2 = cx^2(x-L)^2$$

where $c = -\frac{W}{24EI}$ is a negative constant and $0 \leq x \leq L$. We sketch

$$f(x) = cx^2(x-L)^2 \text{ for } c = -1. f(0) = f(L) = 0.$$

$$f'(x) = cx^2[2(x-L)] + (x-L)^2(2cx) = 2cx(x-L)[x + (x-L)] = 2cx(x-L)(2x-L). \text{ So for } 0 < x < L,$$

$f'(x) > 0 \Leftrightarrow x(x-L)(2x-L) < 0$ (since $c < 0$) $\Leftrightarrow L/2 < x < L$ and $f'(x) < 0 \Leftrightarrow 0 < x < L/2$. So f is increasing on $(L/2, L)$ and decreasing on $(0, L/2)$, and there is a local and absolute minimum at

$$(L/2, f(L/2)) = (L/2, cL^4/16).$$

$$f'(x) = 2c[x(x-L)(2x-L)] \Rightarrow$$

$$f''(x) = 2c[1(x-L)(2x-L) + x(1)(2x-L) + x(x-L)(2)] = 2c(6x^2 - 6Lx + L^2) = 0 \Leftrightarrow$$

$$x = \frac{6L \pm \sqrt{12L^2}}{12} = \frac{1}{2}L \pm \frac{\sqrt{3}}{6}L, \text{ and these are the } x\text{-coordinates of the two inflection points.}$$

