

Mathematics 1a, Section 4.5 Solutions

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1. **a.** $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$.
 - b.** $\lim_{x \rightarrow a} \frac{f(x)}{p(x)} = 0$ because the numerator approaches 0 while the denominator becomes large.
 - c.** $\lim_{x \rightarrow a} \frac{h(x)}{p(x)} = 0$ because the numerator approaches a finite number while the denominator becomes large.
 - d.** If $\lim_{x \rightarrow a} p(x) = \infty$ and $f(x) \rightarrow 0$ through positive values, then $\lim_{x \rightarrow a} \frac{p(x)}{f(x)} = \infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x^2$]. If $f \rightarrow 0$ through negative values, then $\lim_{x \rightarrow a} \frac{p(x)}{f(x)} = -\infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = -x^2$]. If $f(x) \rightarrow 0$ through both positive and negative values, then the limit might not exist. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x$]. It is not possible to evaluate this limit.
 - e.** $\lim_{x \rightarrow a} \frac{p(x)}{q(x)}$ is an indeterminate form of type $\frac{\infty}{\infty}$.
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4. **a.** $\lim_{x \rightarrow a} [f(x)]^{g(x)}$ is an indeterminate form of type 0^0 .
 - b.** If $y = [f(x)]^{p(x)}$, then $\ln y = p(x) \ln f(x)$. When x is near a , $p(x) \rightarrow \infty$ and $\ln f(x) \rightarrow -\infty$, so $\ln y \rightarrow -\infty$. Therefore, $\lim_{x \rightarrow a} [f(x)]^{p(x)} = \lim_{x \rightarrow a} y = \lim_{x \rightarrow a} e^{\ln y} = 0$, provided f^p is defined.
 - c.** $\lim_{x \rightarrow a} [h(x)]^{p(x)}$ is an indeterminate form of type 1^∞ .
 - d.** $\lim_{x \rightarrow a} [p(x)]^{f(x)}$ is an indeterminate form of type ∞^0 .
 - e.** If $y = [p(x)]^{q(x)}$, then $\ln y = q(x) \ln p(x)$. When x is near a , $q(x) \rightarrow \infty$ and $\ln p(x) \rightarrow \infty$, so $\ln y \rightarrow \infty$. Therefore, $\lim_{x \rightarrow a} [p(x)]^{q(x)} = \lim_{x \rightarrow a} y = \lim_{x \rightarrow a} e^{\ln y} = \infty$.
 - f.** $\lim_{x \rightarrow a} \sqrt[q(x)]{p(x)} = \lim_{x \rightarrow a} [p(x)]^{1/q(x)}$ is an indeterminate form of type ∞^0 .
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- 8.

$$\lim_{x \rightarrow 0} \frac{x + \tan x}{\sin x} = \lim_{x \rightarrow 0} \frac{1 + \sec^2 x}{\cos x} = \frac{1 + 1^2}{1} = 2$$

The first equality is by L'Hôpital's Rule. Note that this is the correct spelling, not L'Hospital, as in the book.

12.

$$\lim_{x \rightarrow \infty} \frac{\ln \ln x}{x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{\ln x} \frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{1}{x \ln x} = 0$$

The first equality is by L'Hôpital's Rule.

28.

$$\begin{aligned} \lim_{x \rightarrow 0} (\csc x - \cot x) &= \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{\cos x} = 0 \end{aligned}$$

The first equality on the last line is by L'Hôpital's Rule.

33.

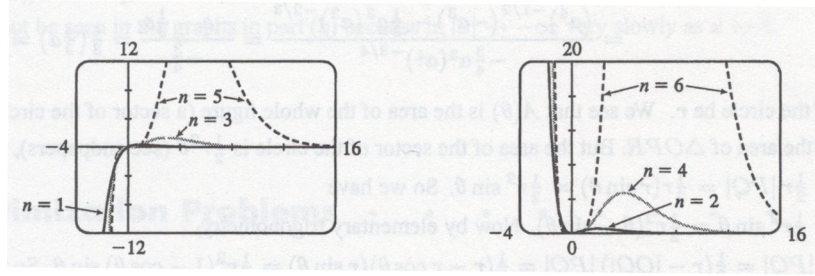
$$\begin{aligned} y &= (1 - 2x)^{1/x} \\ \ln y &= \frac{1}{x} \ln(1 - 2x) \\ \lim_{x \rightarrow 0} \ln y &= \lim_{x \rightarrow 0} \frac{\ln(1 - 2x)}{x} = \lim_{x \rightarrow 0} \frac{-2/(1 - 2x)}{1} = -2 \\ \lim_{x \rightarrow 0} (1 - 2x)^{1/x} &= \lim_{x \rightarrow 0} e^{\ln y} = \lim_{x \rightarrow 0} e^{-2} \end{aligned}$$

The second-to-last equality on the second-to-last line is by L'Hôpital's Rule.

34.

$$\begin{aligned} y &= \left(1 + \frac{a}{x}\right)^{bx} \\ \ln y &= bx \ln \left(1 + \frac{a}{x}\right) \\ \lim_{x \rightarrow \infty} \ln y &= \lim_{x \rightarrow \infty} \frac{b \ln(1 + a/x)}{1/x} = \lim_{x \rightarrow \infty} \frac{b \left(\frac{1}{1+a/x}\right) \left(-\frac{a}{x^2}\right)}{-1/x^2} = \lim_{x \rightarrow \infty} \frac{ab}{1 + a/x} = ab \\ \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^{bx} &= \lim_{x \rightarrow \infty} e^{\ln y} = e^{ab} \end{aligned}$$

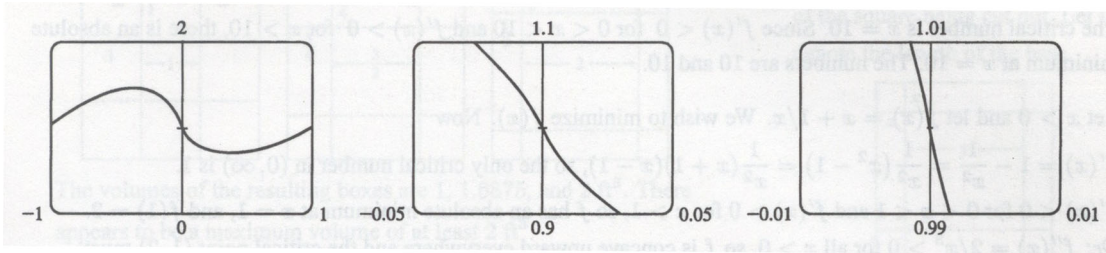
50.



The first figure shows representative examples of $f(x) = x^n e^{-x}$ with n odd. n is even in the second figure. All curves pass through the origin and approach $y = 0$ and $x \rightarrow \infty$. $f'(x) = \frac{x^n(n-x)}{x e^x} = 0$, which is equivalent to $x = n$ or $x = 0$ (the latter for $n > 1$). At $x = 0$, we have a local minimum for n even. At $x = n$, we have a local maximum for all n . As n increases, $(n, f(n))$ gets farther away from the origin. $f''(x) = \frac{x^n(x^2 - 2nx + n^2 - n)}{x^2 e^x} = 0$ is equivalent to $x = n \pm \sqrt{n}$ or $x = 0$ (the latter for $n > 2$). As n increases, the inflection points move farther away from the origin - they are symmetric about the line $x = n$.

58. a. For f to be continuous, we need $\lim_{x \rightarrow 0} f(x) = f(0) = 1$. We note that for $x \neq 0$, $\ln f(x) = \ln |x|^x = x \ln |x|$. So $\lim_{x \rightarrow 0} \ln f(x) = \lim_{x \rightarrow 0} x \ln |x| = \lim_{x \rightarrow 0} \frac{\ln |x|}{1/x} = \lim_{x \rightarrow 0} \frac{1/x}{-1/x^2} = 0$ (the second-to-last equality is by L'Hôpital's Rule). Therefore, $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} e^{\ln f(x)} = e^0 = 1$. So f is continuous at 0.

b. From the graphs, it appears that f is differentiable at 0.



c. To find f' , we use logarithmic differentiation:

$$\begin{aligned} \ln f(x) &= x \ln |x| \\ \frac{f'(x)}{f(x)} &= x \left(\frac{1}{x} \right) + \ln |x| \\ f'(x) &= f(x)(1 + \ln |x|) = |x|^x (1 + \ln |x|) \end{aligned}$$

for $x \neq 0$. Now $f'(x) \rightarrow -\infty$ as $x \rightarrow 0$ (since $|x|^x \rightarrow 1$ and $(1 + \ln |x|) \rightarrow -\infty$), so the curve has a vertical tangent at $(0, 1)$ and is therefore not differentiable there. The fact cannot be seen in the graphs in part **b** because $\ln |x| \rightarrow -\infty$ very slowly as $x \rightarrow 0$.