Mathematics 1a, Section 4.4 Solutions

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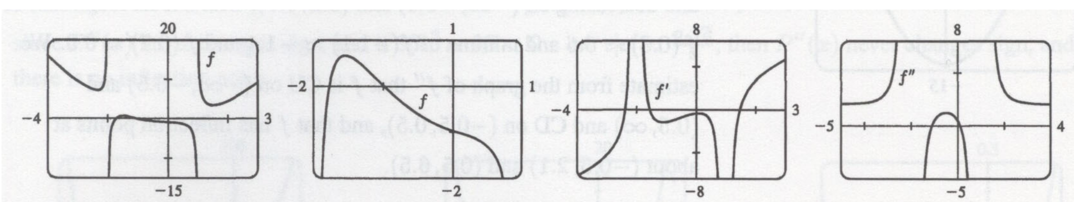
December 5, 2004

4.

$$f(x) = \frac{x^4 + x^3 - 2x^2 + 2}{x^2 + x - 2}$$
$$f'(x) = 2\frac{x^5 + 2x^4 - 3x^3 - 4x^2 + 2x - 1}{(x^2 + x - 2)^2}$$
$$f''(x) = 2\frac{x^6 + 3x^5 - 3x^4 - 11x^3 + 12x^2 + 18x - 2}{(x^2 + x - 2)^3}$$

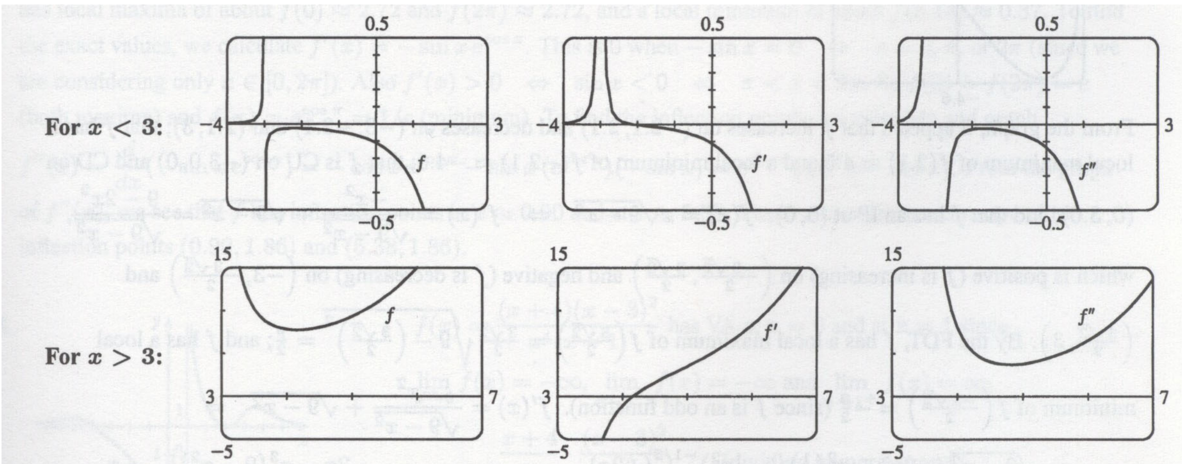


We estimate from the graph of f' that f is increasing on $(-2.4, -2)$, $(-2, -1.5)$, and $(1.5, \infty)$ and decreasing on $(-\infty, -2.4)$, $(-1.5, 1)$, and $(1, 1.5)$. The local maximum is $f(-1.5) \approx 0.7$, and the two local minima are $f(-2.4) \approx 7.2$ and $f(1.5 \approx 3.4$. From the graph of f'', we estimate that f is concave up on $(-\infty, -2)$, $(-1.1, 0.1)$, and $(1, \infty)$ and concave down on $(-2, -1.1)$ and $(0.1, 1)$. f has inflection points at $(-1.1, 0.2)$ and $(0.1, -1.1)$.

8.

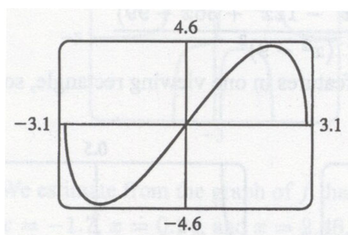
$$f(x) = \frac{e^x}{x^2 - 9}$$
$$f'(x) = \frac{e^x(x^2 - 2x - 9)}{(x^2 - 9)^2}$$
$$f''(x) = \frac{e^x(x^4 - 4x^3 - 12x^2 + 36x + 99)}{(x^2 - 9)^3}$$

There are vertical asymptotes at $x = \pm 3$. It is difficult to show all the pertinent viewing features in one viewing rectangle, so we'll show f, f', f'' for $x < 3$ and $x > 3$ separately.



We estimate from the graphs of f' and f that f is increasing on $(-\infty, -3)$, $(-3, -2.6)$, and $(4.16, \infty)$ and decreasing on $(-2.16, 3)$ and $(3, 4.16)$. There is a local maximum at $f(-2.16) \approx -0.03$ and a local minimum at $f(4.16) \approx 7.71$. From the graphs of f'', we see that f is concave up on $(-\infty, -3)$ and $(3, \infty)$ and concave down on $(-3, 3)$. There is no inflection point.

10.



From the graph, it appears that f increases on $(-2.1, 2.1)$ and decreases on $(-3, -2.1)$ and $(2.1, 3)$; that f has a local maximum of $f(2.1) \approx 4.5$ and a local minimum of $f(-2.1) \approx -4.5$; that f is concave up on $(-3.0, 0)$ and concave down on $(0, 3.0)$, and that f has an inflection point at $(0, 0)$.

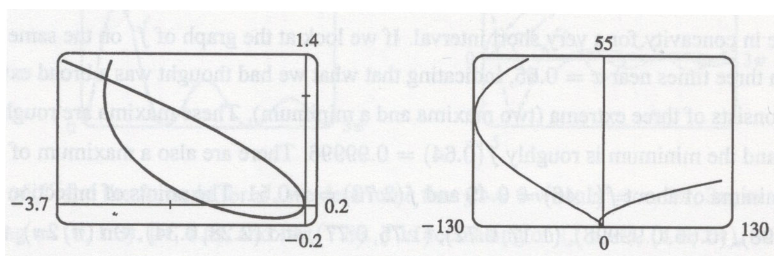
$$f(x) = x\sqrt{9 - x^2}$$
$$f'(x) = \frac{-x^2}{\sqrt{9-x^2}} + \sqrt{9-x^2} = \frac{9-2x^2}{\sqrt{9-x^2}}$$

f' is positive (f is increasing) on $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$ and negative (f is decreasing) on $\left(-3, -\frac{3\sqrt{2}}{2}\right)$ and $\left(\frac{3\sqrt{2}}{2}, 3\right)$. By the first derivative test, f has a local maximum of $f\left(\frac{3\sqrt{2}}{2}\right) = \frac{9}{2}$ and a local minimum of $f\left(-\frac{3\sqrt{2}}{2}\right) = -\frac{9}{2}$.

$$
\begin{aligned}
f'(x) &= \frac{-x^2}{\sqrt{9-x^2}} + \sqrt{9-x^2} \\
f''(x) &= \frac{\sqrt{9-x^2}(-2x) + x^2(\frac{1}{2})(9-x^2)^{-1/2}(-2x)}{9-x^2} - x(9-x^2)^{-1/2} \\
&= \frac{-2x - x^3(9-x^2)^{-1} - x}{\sqrt{9-x^2}} \\
&= \frac{-3x}{\sqrt{9-x^2}} - \frac{x^3}{(9-x^2)^{3/2}} \\
&= \frac{x(2x^2-27)}{(9-x^2)^{3/2}}
\end{aligned}
$$

which is positive (f is concave up) on $(-3,0)$ and negative (f is concave down) on $(0,3)$. f has inflection point at $(0,0)$.

22.

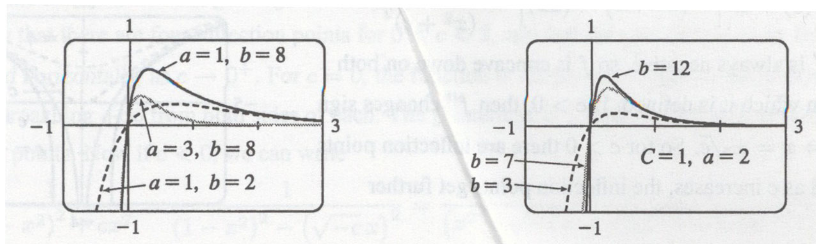


We graph the curve $x = t^4 + 4t^4 - 8t^2$, $y = 2t^2 - t$ in the viewing rectangle $[-3.7, 0.2]$ by $[-0.2, 1.4]$. It appears that there is a horizontal tangent at about $(-0.4, -0.1)$, and vertical tangents at about $(-3,1)$ and $(0,0)$. We calculate

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t-1}{4t^3 + 12t^2 - 16t}$$

so there is a horizontal tangent where $dy/dt = 4t - 1 = 0$, so $t = 1/4$. This point (the lowest point) is shown in the first graph. Its coordinate are $(-\frac{111}{256}, -\frac{1}{8})$. There are vertical tangents where $dx/dt = 4t^3 + 12t^2 - 16t = 0$, which means $4t(t+4)(t-1) = 0$. We have missed one vertical tangent corresponding to $t = -4$, and if we plot the graph for $t \in [-5, 3]$, we see that the curve has another vertical tangent line at approximately $(-128, 36)$. The t-values and points at which there are vertical tangents are $t = 0$, $(0,0)$; $t = -4$, $(-128, 36)$; and $t = 1$, $(-3, 1)$.

24. For $f(t) = C(e^{-at} - e^{-bt})$, C affects only vertical stretching, so we let $C = 1$. From the first figure, we notice that the graphs all pass through the origin, approach the t-axis as t increases, and approach $-\infty$ as $t \to -\infty$. Next we let $a = 2$ and produce the second figure.



Here, as b increases, the slope of the tangent at the origin increases and the local maximum value increases. $f(t) = e^{-2t} - e^{-bt}$, so $f'(t) = be^{-bt} - 2e^{-2t}$. So $f'(0) = b - 2$, which increases as b increases. If $f'(t) = 0$, then $be^{-bt} = 2e^{-2t}$, so $\frac{b}{2} = e^{(b-2)t}$, so $\ln \frac{b}{2} = (b-2)t$, so $t = t_1 = \frac{\ln b - \ln 2}{b-2}$, which decreases as b increases (the maximum is getting closer to the y-axis). $f(t_1) = \frac{(b-2)2^{2/(b-2)}}{b^{1+2/(b-2)}}$. We can show that this value increases as b increases by considering it to be a function of b and graphing its derivative with respect to b, which is always positive.

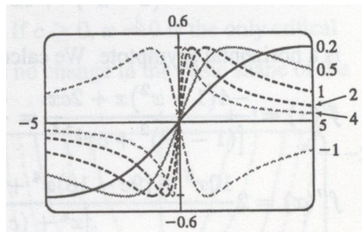
25. Note that $c = 0$ is a transitional value at which the graph consists of the x-axis. Also, we can see that if we substitute $-c$ for c, the function $f(x) = \frac{cx}{1+c^2x^2}$ will be reflected in the x-axis, so we investigate only positive values of c (except $c = -1$ as a demonstration of this reflective property). Also, f is an odd function. $\lim_{x\to\pm\infty} f(x) = 0$, so $y = 0$ is a horizontal asymptote for all c. We calculate

$$f'(x) = \frac{(1+c^2x^2)c - cx(2c^2x)}{(1+c^2x^2)^2} = -\frac{c(c^2x^2-1)}{(1+c^2x^2)^2}$$

So $f'(x) = 0$ if and only if $c^2x^2 - 1 = 0$, so $x = \pm 1/c$. So there is an absolute maximum of $f(1/c) = 1/2$ and an absolute minimum of $f(-1/c) = -1/2$. These extrema have the same value regardless of c, but the maximum points move closer to the y-axis as c increases.

$$\begin{aligned} f''(x) &= \frac{(-2c^3x)(1+c^2x^2)^2 - (-c^3x^2+c)[2(1+c^2x^2)(2c^2x)]}{(1+c^2x^2)^4} \\ &= \frac{(-2c^3x)(1+c^2x^2) + (c^3x^2-c)(4c^2x)}{(1+c^2x^2)^3} \\ &= \frac{2c^3x(c^2x^2-3)}{(1+c^2x^2)^3} \end{aligned}$$

We see that $f''(x) = 0$ when $x = 0$ or $x = \pm\sqrt{3}/c$, so there are inflection points at $(0, 0)$ and at $(\pm\sqrt{3}/c, \pm\sqrt{3}/4)$. Again, the y-coordinate of the inflection points does not depend on c, but as c increases, both inflection points approach the y-axis.



32. a.

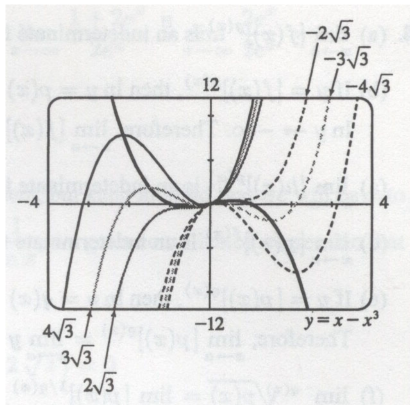
$$f(x) = 2x^3 + cx^2 + 2x$$
$$f'(x) = 6x^2 + 2cx + 2 = 2(3x^2 + cx + 1)$$

So we see $f'(x) = 0$ means $x = \frac{-c\pm\sqrt{c^2-12}}{6}$. So f has critical points when $c^2 - 12 \geq 0$, that is, when $|c| \geq 2\sqrt{3}$. For $c = \pm 2\sqrt{3}$, $f'(x) \geq 0$ on $(-\infty, \infty)$, so f' does not change signs at $-c/6$, and there is no extremum. If $c^2 - 12 > 0$, then f' changes from positive to negative at $x = \frac{-c-\sqrt{c^2-12}}{6}$ and from negative to positive at $x = \frac{-c+\sqrt{c^2-12}}{6}$. So f has a local maximum at $x = \frac{-c-\sqrt{c^2-12}}{6}$ and a local minimum at $x = \frac{-c+\sqrt{c^2-12}}{6}$.

b. Let x_0 be a critical number for $f(x)$. Then $f'(x_0) = 0$, so $3x_0^2 + cx_0 + 1 = 0$, so $c = \frac{-1-3x_0^2}{x_0}$. Now

$$\begin{aligned} f(x_0) &= 2x_0^3 + cx_0^2 + 2x_0 \\ &= 2x_0^3 + x_0^2\left(\frac{-1-3x_0^2}{x_0}\right) + 2x_0 \\ &= 2x_0^3 - x_0 - 3x_0^3 + 2x_0 \\ &= x_0 - x_0^3 \end{aligned}$$

So the point is $(x_0, y_0) = (x_0, x_0 - x_0^3)$, that is, the point lies on the curve $y = x - x^3$.



$-2\sqrt{3}$
$-3\sqrt{3}$
$-4\sqrt{3}$
12
-4
4
-12
$y = x - x^3$
$4\sqrt{3}$
$3\sqrt{3}$
$2\sqrt{3}$