Mathematics 1a, Section 4.3 Solutions

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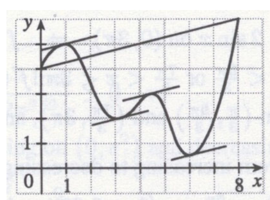
1.

$$\frac{f(8)-f(0)}{8-0} = \frac{6-4}{8} = \frac{1}{4}$$

The values of c which satisfy $f'(c) = 1/4$ seem to be about $c = 0.8, 3.2, 4.4$, and 6.1.



2. a. g is concave upward on $(-1, 2)$ and $(7, 8)$.

b. g is concave downward on $(2, 4)$ and $(4, 7)$.

c. The only point of inflection is $(2, 2)$. Note that 7 is not in the domain of this function.

4. a. See the First Derivative Test.

b. See the Second Derivative Test and the note that precedes Example 5.

6. a. f is increasing on the intervals where $f'(x) > 0$, namely, $(2, 4)$ and $(6, 9)$.

b. f has a local maximum where it changes from increasing to decreasing, that is, where f' changes from positive to negative (at $x = 4$). Similarly, where f' changes from negative to positive, f has a local minimum (at $x = 2$ and at $x = 6$).

c. When f' is increasing, its derivative f'' is positive and hence, f is concave upward. This happens on $(1, 3)$, $(5, 7)$, and $(8, 9)$. Similarly, f is concave downward when f' is decreasing - that is, on $(0, 1)$, $(3, 5)$, and $(7, 8)$.

d. f has inflection points at $x = 1, 3, 5, 7, 8$, since the direction of concavity changes at each of these values.

8. a. $f(x) = 1 + 8x - x^8$, thus $f'(x) = 8 - 8x^7 = 8(1 - x^7)$. Thus, $f'(x) = 1 - x^7 > 0$ means that $x^7 < 1$, so $x < 1$, and $f'(x) < 0$ when $x > 1$. So f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$.

b. f changes from increasing to decreasing at $x = 1$. Thus, $f(1) = 8$ is a local maximum.

c. $f''(x) = -56x^6$. $f''(x) < 0$ for any $x \neq 0$, so f is concave downward on $(-\infty, 0)$ and $(0, \infty)$ by the Concavity Test. In fact, f is concave down on $\mathbb{R}$ because f' is decreasing on $\mathbb{R}$. There are no inflection points.

10. a. $f(x) = x/(1 + x)^2$, so we compute

$$f'(x) = \frac{(1+x)^2(1) - (x)2(1+x)}{((1+x)^2)^2} = \frac{(1+x)((1+x) - 2x)}{(1+x)^4} = \frac{(1+x)(1-x)}{(1+x)^4} = \frac{1-x}{(1+x)^3}$$

So $f'(x) > 0$ when $-1 < x < 1$ and $f'(x) < 0$ when $x < -1$ or $x > 1$. So f is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1)$ and $(1, \infty)$.

b. f changes from increasing to decreasing at $x = 1$. The point $x = -1$ is not in the domain of f. Thus, $f(1) = 1/4$ is a local maximum.

c. We compute

$$f''(x) = \frac{(1+x)^3(-1) - (1-x)3(1+x)^2}{((1+x)^3)^2} = \frac{(1+x)^2(-1(1+x) - 3(1-x))}{(1+x)^6} = \frac{2x-4}{(1+x)^4}$$

Thus $f''(x) > 0$ when $x > 2$ and $f''(x) < 0$ when $x < 2$ (or course, $x \neq -1$). Thus, f is concave upward on $(2, \infty)$ and f is concave downward on $(-\infty, -1)$ and $(-1, 2)$. There is an inflection point at $(2, 2/9)$.

14. a.

$$y = f(x) = x \ln x$$
$$f'(x) = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x$$

Thus $f'(x) > 0$ when $\ln x + 1 > 0$, that is, when $x > e^{-1}$. Therefore f is increasing on $(1/e, \infty)$ and decreasing on $(0, 1/e)$.

b. f changes from decreasing to increasing at $x = 1/e$, so $f(1/e) = -1/e$ is a local minimum.

c. $f''(x) = 1/x > 0$ for $x > 0$. So f is concave upward on its entire domain, and has no inflection point.

18. a.

$$g(x) = 200 + 8x^3 + x^4$$
$$g'(x) = 24x^2 + 4x^3 = 4x^2(6 + x)$$

We see that $g'(x) = 0$ when $x = -6$ and when $x = 0$. $g'(x) > 0$ when $x > -6$ (and $x \neq 0$), and $g'(x) < 0$ when $x < -6$, so g is decreasing on $(-\infty, -6)$ and increasing on $(-6, \infty)$ with a horizontal tangent at $x = 0$.

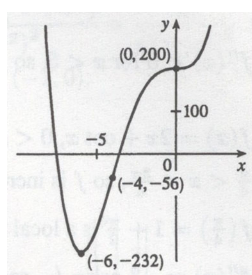
b. $g(-6) = -232$ is a local minimum value. There is no local maximum value.

c.

$$g''(x) = 48x + 12x^2 = 12x(4 + x)$$

So we see $g''(x) = 0$ when $x = -4$ and when $x = 0$. $g''(x) > 0$ when $x < -4$ or $x > 0$, and $g''(x) < 0$ when $-4 < x < 0$, so g is concave up on $(-\infty, -4)$ and $(0, \infty)$, and g is concave down on $(-4, 0)$. Inflection points are at $(-4, -56)$ and $(0, 200)$.

d.



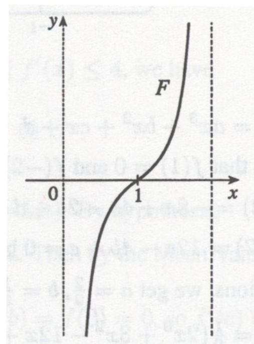
38.

$$F(x) = -\frac{k}{x^2} + \frac{k}{(x-2)^2}$$

Where $k > 0$ and $0 < x < 2$. For $0 < x < 2$, $x - 2 < 0$, so

$$F'(x) = \frac{2k}{x^3} - \frac{2k}{(x-2)^3}$$

is positive and thus F is increasing. $\lim_{x \to 0^+} F(x) = -\infty$ and $\lim_{x \to 0^-} F(x) = \infty$, so $x = 0$ and $x = 2$ are vertical asymptotes. Notice that when the middle particle is at $x = 1$, the net force acting on it is 0. When $x > 1$, the net force is positive, meaning that it acts to the right. And if the particle approaches $x = 2$, the force on it rapidly becomes very large. When $x < 1$, the net force is negative, so it acts to the left. If the particle approaches 0, the force becomes very large to the left.



43. $f(x) = \tan x - x$, so $f'(x) = \sec^2 x - 1$, which is positive for $0 < x < \pi/2$, since $\sec^2 x > 1$ for $0 < x < \pi/2$. So f is increasing on $(0, \pi 2)$. Thus, $f(x) > f(0)$ for $0 < x < \pi/2$, thus $\tan x - x > 0$, thus $\tan x > x$ for $0 < x < \pi/2$.

44. a. Let $f(x) = e^x - (1 + x) = e^x - 1 - x$. Now $f(0) = e^0 - 1 = 0$, and for $x \geq 0$, we have $f'(x) = e^x - 1 \geq 0$. Now, since $f(0) = 0$ and f is increasing on $[0, \infty)$, $f(x) \geq 0$ for $x \geq 0$, thus $e^x - 1 - x \geq 0$, thus $e^x \geq 1 + x$.

b. We proceed analogously to part **a**. Let $f(x) = e^x - 1 - x - \frac{1}{2}x^2$. Thus $f'(x) = e^x - 1 - x$, which by part **a**, is positive for $x \geq 0$. Thus $f(x)$ is increasing on $(0, \infty)$, so on that interval, $0 = f(0) \leq f(x) = e^x - 1 - x - \frac{1}{2}x^2$, so $e^x \geq 1 + x + \frac{1}{2}x^2$.

c. We will use mathematical induction to generalize the results of parts **a** and **b**. First, however, a quick review of mathematical induction: if we wish to prove a statement is true for any n, we need to prove two things. First, we must prove it holds true for $n = 1$. Then we must prove that if it holds for a particular k, then it holds for the next k, i.e., $k + 1$. The intuition behind this is that we've proven the $n = 1$ case. Next, the $k \Rightarrow k + 1$ for $n = 1$ implies the $n = 2$ case. The $n = 2$ case similarly implies the $n = 3$ case, and so on, for all n.

In part **a** we proved the $n = 1$ case, so we need to show that if the statement is true for k, it is true for $k + 1$. The essential logic is the same we used in part **b**. Set

$$f(x) = e^x - 1 - x - \frac{1}{2}x^2 - \ldots - \frac{1}{(k+1)!}x^{k+1}$$
$$f'(x) = e^x - 1 - x - \frac{1}{2}x^2 - \ldots - \frac{1}{k!}x^k$$

In the second line we have used the fact that $(n+1)!/n! = n+1$ (check the derivative for $k = 1, 2, 3$ to see why this is true, if you're not convinced). However, our assumption that the statement is true for k tells us that $f'(x) \geq 0$ on $(0, \infty)$, since $f'(x)$ for our $k+1$ case is simply the corresponding $f(x)$ for the k case. Since $f'(x) \geq 0$, we know $f(x) \geq 0$ on $(0, \infty)$, since $f(0) = 0$. Thus:

$$\begin{aligned} f(x) = e^x - 1 - x - \frac{1}{2}x^2 - \ldots - \frac{1}{(k+1)!}x^{k+1} &\geq 0 \\ e^x \geq 1 + x + \frac{1}{2}x^2 + \ldots + \frac{1}{(k+1)!}x^{k+1} \end{aligned}$$

So, by mathematical induction, $e^x \geq 1 + x + \frac{1}{2}x^2 + \ldots + \frac{1}{n!}x^n$ for all positive integers n.

46. By the Mean Value Theorem,

$$\frac{f(5) - f(2)}{5 - 2} = f'(c)$$

for some $c \in (2, 5)$. Since $1 \leq f'(x) \leq 4$, we have

$$\begin{aligned} 1 \leq \frac{f(5) - f(2)}{5 - 2} &\leq 4 \\ 1 \leq \frac{f(5) - f(2)}{3} &\leq 4 \\ 3 \leq f(5) - f(2) &\leq 12 \end{aligned}$$

48. Let $v(t)$ be the velocity of the car t hours after 2pm. Remember that 10 minutes are 1/6 of an hour. Then

$$\frac{v(1/6) - v(0)}{1/6 - 0} = \frac{50 - 30}{1/6} = 120$$

By the Mean Value Theorem, there is a number c such that $0 < c < \frac{1}{6}$ with $v'(c) = 120$. Since $v'(t)$ is the acceleration at time t, the acceleration c hours after 2pm is exactly 120 mi/h^2.