

# Mathematics 1a, Section 3.6 Solutions

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**1. a.**

$$\begin{aligned}\frac{d}{dx}(xy + 2x + 3x^2) &= \frac{d}{dx}(4) \\ (x \cdot y' + y \cdot 1) + 2 + 6x &= 0 \\ xy' &= -y - 2 - 6x \\ y' &= \frac{-y - 2 - 6x}{x} = -6 - \frac{y + 2}{x}\end{aligned}$$

**b.**

$$\begin{aligned}xy + 2x + 3x^2 &= 4 \\ xy &= 4x - 2x - 3x^2 \\ y &= \frac{4 - 2x - 3x^2}{x} = \frac{4}{x} - 2 - 3x \\ y' &= -\frac{4}{x^2} - 3\end{aligned}$$

**c.** Using part **a.** we get

$$y' = \frac{-y - 2 - 6x}{x} = \frac{-(4/x - 2 - 3x) - 2 - 6x}{x} = \frac{-4/x - 3x}{x} = -\frac{4}{x^2} - 3$$

**2. a.**

$$\begin{aligned}\frac{d}{dx}(4x^2 + 9y^2) &= \frac{d}{dx}(36) \\ 8x + 18yy' &= 0 \\ y' &= -\frac{8x}{18y} = -\frac{4x}{9y}\end{aligned}$$

b.

$$4x^2 + 9y^2 = 36$$

$$9y^2 = 36 - 4x^2$$

$$y = \frac{4}{9}(9 - x^2)$$

$$y = \pm \frac{2}{3}\sqrt{9x^2}$$

$$y' = \pm \frac{2}{3} \cdot \frac{1}{2}(9 - x^2)^{-1/2}(-2x) = \mp \frac{2x}{3\sqrt{9 - x^2}}$$

c. From part a.,

$$y' = -\frac{4x}{9y} = -\frac{4x}{9(\pm \frac{2}{3}\sqrt{9 - x^2})} = \mp \frac{2x}{3\sqrt{9 - x^2}}$$

6.

$$\frac{d}{dx}(y^5 + x^2y^3) = \frac{d}{dx}(1 + ye^{x^2})$$

$$5y^4y' + (x^2 \cdot 3y^2y' + y^3 \cdot 2x) = 0 + y \cdot e^{x^2}(2x) + e^{x^2} \cdot y'$$

$$y'(5y^4 + 3x^2y^2 - e^{x^2}) = 2xye^{x^2} - 2xy^3$$

$$y' = \frac{2xy(e^{x^2} - y^2)}{5y^4 + 3x^2y^2 - e^{x^2}}$$

14.

$$\frac{x^2}{9} + \frac{y^2}{36} = 1$$

$$\frac{2x}{9} + \frac{yy'}{18} = 0$$

$$y' = -\frac{4x}{y}$$

When  $x = -1$  and  $y = 4\sqrt{2}$  we have

$$y' - \frac{4(-1)}{4\sqrt{2}} = \frac{1}{\sqrt{2}}$$

so an equation of the tangent line is

$$y = \frac{1}{\sqrt{2}}(x + 9)$$

18.

$$\begin{aligned}x^2y^2 &= (y+1)^2(4-y^2) \\2x^2yy' + 2xy^2 &= (y+1)^2(-2yy') + (4-y^2) \cdot 2(y+1)y' \\xy^2 &= [(y+1)(4-y^2) - y(y+1)^2 - x^2y] y' \\y' &= \frac{xy^2}{(y+1)(4-y^2) - y(y+1)^2 - x^2y} = 0\end{aligned}$$

when  $x = 0$ . So an equation of the tangent line at  $(0, -2)$  is

$$y = -2$$

20. a.

$$\begin{aligned}y^2 &= x^3 + 3x^2 \\2yy' &= 3x^2 + 3(2x) \\y' &= \frac{3x^2 + 6x}{2y}\end{aligned}$$

So at the point  $(1, -2)$  we have

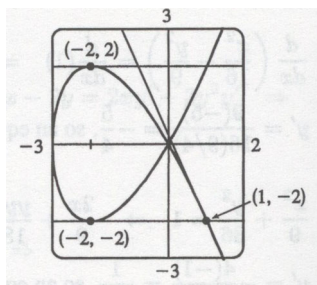
$$y' = \frac{3(1)^2 + 6(1)}{2(-2)} = -\frac{9}{4}$$

and an equation of the tangent line is

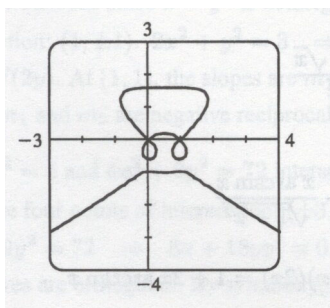
$$y = -\frac{9}{4}x + \frac{1}{4}$$

**b.** The curve has a horizontal tangent where  $y' = 0$ , or equivalently,  $3x^2 + 6x = 0$ , which occurs when  $x = 0$  or  $x = -2$ . But note that at  $x = 0$ ,  $y = 0$  also, so the derivative does not exist. At  $x = -2$ ,  $y^2 = (-2)^3 + 3(-2)^2 = -8 + 12 = 4$ , so  $y = \pm 2$ . So the two points at which the curve has a horizontal tangent are  $(-2, -2)$  and  $(-2, 2)$ .

**c.**



22. a.



b. There are 0 points with horizontal tangents: 3 at  $x = 0$ , 3 at  $x = \frac{1}{2}$ , and 3 at  $x = 1$ . The three horizontal tangents along the top of the wagon are hard to find, but by limiting the  $y$ -range of the graph (for example, to  $[1.6, 1.7]$ ) they are distinguishable.

26.

$$x^2 + 6xy + y^2 = 8$$

$$2x + 6xy' + 6y + 2yy' = 0$$

$$(3x + y)y' = -x - 3y$$

$$y' = -\frac{x + 3y}{3x + y}$$

$$\begin{aligned} y'' &= -\frac{(3x + y)(1 + 3y') - (x + 3y)(3 + y')}{(3x + y)^2} \\ &= -\frac{-8y + 8xy'}{(3x + y)^2} = \frac{8(y - xy')}{(3x + y)^2} \\ &= \frac{8[y - x(-x - 3y)/(3x + y)]}{(3x + y)^2} \cdot \frac{3x + y}{3x + y} \\ &= \frac{8[y(3x + y) + x(x + 3y)]}{(3x + y)^2} \\ &= \frac{8(x^2 + 6xy + y^2)}{(3x + y)^3} \\ &= \frac{64}{(3x + y)^3} \end{aligned}$$

At the last step, we used the fact that  $x$  and  $y$  must satisfy the original equation,  $x^2 + 6xy + y^2 = 8$ .

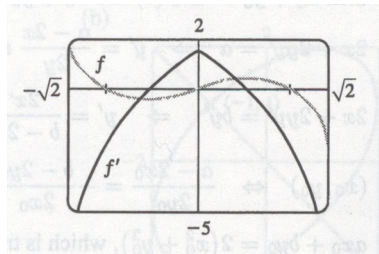
36. Note that  $\arcsin x = \sin^{-1} x$ , which we call “arc-sine,” the inverse function for  $\sin x$ .  $\sin^{-1} x$  is *NOT*  $\frac{1}{\sin x}$  (which can also be written as  $(\sin x)^{-1}$ ), which is the *multiplicative*

inverse of  $\sin x$ . That is:

$$\begin{aligned}\sin(\arcsin x) &= \sin(\sin^{-1} x) = x \\ \arcsin(\sin x) &= \sin^{-1}(\sin x) = x \\ (\sin x)^{-1} \sin x &= \frac{1}{\sin x} \sin x = 1\end{aligned}$$

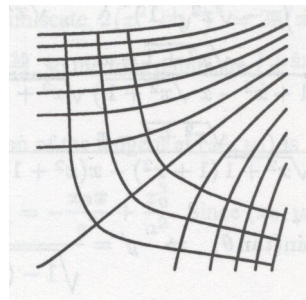
Now we can go on to solve the problem.

$$\begin{aligned}f(x) &= x \arcsin(1 - x^2) \\ f'(x) &= x \left[ \frac{-2x}{\sqrt{1 - (1 - x^2)^2}} \right] + \arcsin(1 - x^2) \cdot 1 \\ f'(x) &= \arcsin(1 - x^2) - \frac{2x^2}{\sqrt{2x^2 - x^4}}\end{aligned}$$



This is reasonable because the graphs show that  $f$  is increasing when  $f'$  is positive, and that  $f$  has an inflection point when  $f'$  changes from increasing to decreasing.

**40.** The orthogonal family represents the direction of the wind. To understand this, think of the wind as trying to get from high to low pressure as quickly as possible. Any path passing through isobars will get from high to low, but the quickest way is to always be orthogonal to the isobars.



**42.** The circles  $x^2 + y^2 = ax$  and  $x^2 + y^2 = by$  intersect at the origin where the tangents are vertical and horizontal. If  $(x_0, y_0)$  is the other point of intersection, then

$$x_0^2 + y_0^2 = ax_0$$

$$x_0^2 + y_0^2 = by_0$$

Now we implicitly differentiate.

$$x^2 + y^2 = ax$$

$$2x + 2yy' = a$$

$$y' = \frac{a - 2x}{2y}$$

$$x^2 + y^2 = by$$

$$2x + 2yy' = by'$$

$$y' = \frac{2x}{b - 2y}$$

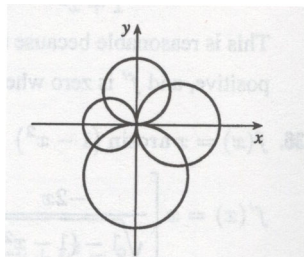
The curves are orthogonal when the two derivatives are negative reciprocals. So this occurs at  $(x_0, y_0)$  if

$$\frac{a - 2x_0}{2y_0} = -\frac{b - 2y_0}{2x_0}$$

$$2ax_0 - 4x_0^2 = 4y_0^2 - 2by_0$$

$$ax_0 + by_0 = 2(x_0^2 + y_0^2)$$

This last line is true because  $(x_0, y_0)$  lies on both circles.



**52. a.**  $f(x) = 2x + \cos x$  implies  $f'(x) = 2 - \sin x$ , which is always positive, since the range of  $\sin x$  is  $[-1, 1]$ . Thus  $f'(x) > 0$  for all  $x$ , so  $f(x)$  is increasing for all  $x$ , and is therefore one-to-one.

**b.** Since  $f$  is one-to-one,  $f^{-1}(1) = k$  means  $f(k) = 1$ . By inspection, we see that this is satisfied for  $k = 0$ :

$$f(k) = f(0) = 2(0) + \cos 0 = 1$$

c.

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{2 - \sin 0} = \frac{1}{2}$$

54.

$$z^2 + 4y^2 = 5$$

$$2x + 4(2yy') = 0$$

$$y' = -\frac{x}{4y}$$

Now let  $h$  be the height of the lamp, and let  $(a, b)$  be the point of tangency of the line passing through the points  $(3, h)$  and  $(-5, 0)$ . This line has slope

$$\frac{h - 0}{3 - (-5)} = \frac{h}{8}$$

But the slope of the tangent line through  $(a, b)$  can be expressed as  $y' = -\frac{a}{4b}$ , or as

$$\frac{b - 0}{a - (-5)} = \frac{b}{a + 5}$$

since the line passes through  $(-5, 0)$  and  $(a, b)$ . So we can equate these two slopes, as they refer to the same line:

$$\begin{aligned} -\frac{a}{4b} &= \frac{b}{a + 5} \\ 4b^2 &= -a - 5a \\ a^2 + 4b^2 &= -5a \end{aligned}$$

But since  $(a, b)$  is on the ellipse, we also have  $a^2 + 4b^2 = 5$ , so  $5 = -5a$ , or  $a = -1$ . Then

$$\begin{aligned} -a^2 - 5a &= -1 - 5(-1) = 4 \\ b &= 1 \end{aligned}$$

since the point is on the top half of the ellipse (so  $b > 0$ ). Thus

$$\begin{aligned} \frac{h}{8} &= \frac{b}{a + 5} = \frac{1}{4} \\ h &= 2 \end{aligned}$$

So the lamp is located 2 units above the  $x$ -axis.