The Substitution Rule : Solutions

① (a) $\int_{-5}^{5} \frac{x^2-2x^4}{x^3+x}\,dx = \boxed{0}$ because $f(x) = \frac{x^2-2x^4}{x^3+x}$

is an <u>odd</u> function. Why? Well,

$$f(-x) = \frac{(-x)^2-2(-x)^4}{(-x)^3+(-x)} = \frac{x^2-2x^4}{-x^3-x}$$

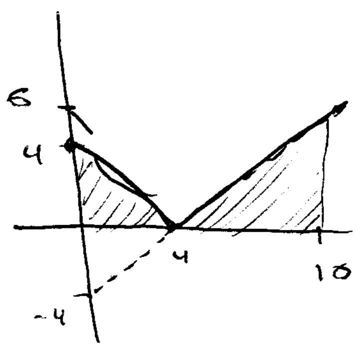
$$= \frac{x^2-2x^4}{-(x^3+x)} = -f(x).$$

Since $f(-x) = -f(x)$, f is an odd function.

(b) $\int_0^{10} |x-4|\,dx = \frac{1}{2}(4)(4) + \frac{1}{2}(6)(6)$

$= 8 + 18$

$= \boxed{26}$



$\int_0^{10} |x-4|\,dx$ = area of 1st triangle + area of 2nd triangle

(c) $\int (6x^2+2)\sin(x^3+x+1)\,dx = \int 2\sin u\,du$

$= -2\cos u + C$

$= \boxed{-2\cos(x^3+x+1)+C}$

$u = x^3+x+1$

$du = (3x^2+1)\,dx$

$2\,du = (6x^2+2)\,dx$

(d) $\int (1+x^3)^{3/2} x^2\,dx = \int u^{3/2}\cdot\frac{1}{3}\,du = \frac{1}{3}\int u^{3/2}\,du$

$= \frac{1}{3}u^{5/2}\cdot\frac{2}{5} + C$

$= \boxed{\frac{2}{15}(1+x^3)^{5/2} + C}$

$u = 1+x^3$

$du = 3x^2\,dx$

$\frac{1}{3}\,du = x^2\,dx$

(e) $\int_{-\pi}^{\pi} x^2 \sin 7x \, dx$

Might this be an odd function?

$$f(-x) = (-x)^2 \sin 7(-x)$$
$$= x^2 \sin(-7x)$$
$$= -x^2 \sin 7x = f(x)$$

because $y = \sin x$ is an odd ftn

Yes, it's an odd ftn, so $\int_{-\pi}^{\pi} x^2 \sin 7x \, dx = \boxed{0}$

(f) $\int x^2 e^{x^3} dx = \frac{1}{3}\int e^u du = \frac{1}{3} e^u + C$

$= \boxed{\frac{1}{3} e^{x^3} + C}$

$u = x^3$

$du = 3x^2 dx$

$\frac{1}{3} du = x^2 dx$

(g) $\int \frac{\tan^{-1} x}{1+x^2} dx = \int u \, du = \frac{1}{2} u^2 + C$

$= \boxed{\frac{1}{2}(\tan^{-1} x)^2 + C}$

$u = \tan^{-1} x$

$du = \frac{1}{1+x^2} dx$

(h) $\int_0^{\pi/4} \tan x \, dx = \int_0^{\pi/4} \frac{\sin x}{\cos x} dx = \int_1^{\sqrt{2}/2} -1/u \, du$

$= -\ln|u| \Big|_1^{\sqrt{2}/2}$

$= -\ln \sqrt{2}/2 + \ln 1$

$= \boxed{-\ln \sqrt{2}/2}$

$u = \cos x \, dx$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

Note that

$x = 0 \Rightarrow$

$u = \cos 0 = 1$

and

$x = \pi/4 \Rightarrow$

$u = \cos \pi/4 = \sqrt{2}/2$

(i) $\int \frac{1+x}{1+x^2}\,dx = \int \left(\frac{1}{1+x^2} + \frac{x}{1+x^2}\right) dx$

$= \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx$

$u = 1+x^2$
$du = 2x\,dx$
$\frac{1}{2}\,du = x\,dx$

$= \arctan x + \frac{1}{2}\int \frac{1}{u}\,du$

$= \arctan x + \frac{1}{2}\ln|u| + C$

$= \boxed{\arctan x + \frac{1}{2}\ln(1+x^2) + C}$

(j) $\int \frac{25}{25+9x^2}\,dx = \int \frac{25}{25+9x^2}\cdot\frac{1/25}{1/25}\,dx$

$= \int \frac{1}{1+\frac{9x^2}{25}}\,dx = \int \frac{1}{1+\left(\frac{3x}{5}\right)^2}\,dx$

$u = \frac{3}{5}x$
$du = \frac{3}{5}\,dx$
$\frac{5}{3}\,du = dx$

$= \frac{5}{3}\int \frac{1}{1+u^2}\,du$

$= \frac{5}{3}\arctan u + C = \boxed{\frac{5}{3}\arctan\left(\frac{3}{5}x\right) + C}$

(k) $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$

$u = 1+2x \Rightarrow u-1 = 2x$
$du = 2\,dx \qquad x = \frac{u-1}{2}$
$\frac{1}{2}\,du = dx$

Note that

x	$u=1+2x$
0	1
4	9

$= \int_1^9 \left(\frac{u-1}{2}\right)\left(\frac{1}{\sqrt{u}}\right)\frac{1}{2}\,du = \frac{1}{4}\int_1^9 \frac{u-1}{\sqrt{u}}\,du$

continued...

$$= \frac{1}{4}\int_1^9 \left(u^{1/2} - u^{-1/2}\right) du$$

$$= \frac{1}{4}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right)\Big|_1^9 = \frac{1}{4}\left(\frac{2}{3}\cdot 9\cdot 3 - 2\cdot 3 - \frac{2}{3} + 2\right)$$

$$= \frac{1}{4}\left(18 - 6 - \frac{2}{3} + 2\right) = \frac{1}{4}\left(13\tfrac{1}{3}\right) = \frac{1}{4}\cdot\frac{40}{3} = \boxed{\frac{10}{3}}$$

(e) $\displaystyle\int_0^1 x\sqrt{1-x^4}\,dx = \int_0^1 x\sqrt{1-(x^2)^2}\,dx$ $\qquad u = x^2$

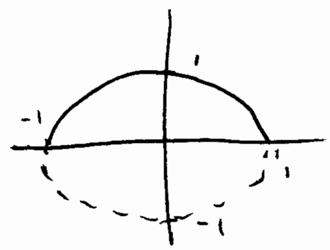
$\qquad du = 2x\,dx$

$\displaystyle = \int_0^1 \frac{1}{2}\sqrt{1-u^2}\,du = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du$ $\qquad \frac{1}{2}du = x\,dx$

$y = \sqrt{1-u^2}$

$y^2 = 1 - u^2$

$u^2 + y^2 = 1$



$y = \sqrt{1-u^2}$ is the top half of the unit circle

$\displaystyle\Rightarrow \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du = \frac{1}{2}\cdot$ area of $\frac{1}{4}$ unit circle

$$= \frac{1}{2}\cdot\frac{1}{4}\cdot\pi(1)^2 = \boxed{\frac{\pi}{8}}$$

(2) $\displaystyle\int_0^{12} g(x)\,dx = \frac{\pi}{12}$ so $\displaystyle\int_0^3 g(4x)\,dx = \frac{1}{4}\int_0^{12} g(u)\,du$

$u = 4x$

$du = 4dx$

$\frac{1}{4}du = dx$

x	$u = 4x$
0	0
3	12

$= \frac{1}{4}\left(\frac{\pi}{12}\right)$

$= \boxed{\frac{\pi}{48}}$

(3) $f(t) = \frac{1}{2}\sin\left(\frac{2\pi t}{5}\right)$ = rate of air flow into lungs at time t

$$\Rightarrow \int_0^t \frac{1}{2}\sin\left(\frac{2\pi x}{5}\right)dx$$ = net change in volume of air in lungs from time 0 to time t

$$u = \frac{2\pi x}{5}$$

$$du = \frac{2\pi}{5}dx$$

$$\frac{5}{2\pi}du = dx$$

$$= \frac{1}{2}\cdot\frac{5}{2\pi}\int_0^{\frac{2\pi t}{5}} \sin u \, du$$

$$= -\frac{5}{4\pi}\cos u \Big|_0^{\frac{2\pi t}{5}}$$

$$= -\frac{5}{4\pi}\left(\cos\left(\frac{2\pi t}{5}\right) - \cos 0\right)$$

$$= -\frac{5}{4\pi}\left(\cos\left(\frac{2\pi t}{5}\right) - 1\right)$$

$$= \boxed{\frac{5}{4\pi}\left(1 - \cos\left(\frac{2\pi t}{5}\right)\right)}$$