Limits and Continuity Solutions

① (a) $0 = f(x) = \dfrac{x^2+2x-3}{x+3} \iff 0 = x^2+2x-3$

$0 = (x+3)(x-1)$

$\boxed{x = -3, 1}$

(b) Since f is a rational ftn, it is continuous on its domain $(-\infty,-3)\cup(-3,\infty)$.

(c) f has a discontinuity at $x=-3$ because $f(-3)$ is undefined.

Since $\displaystyle\lim_{x\to-3} f(x) = \lim_{x\to-3}\frac{x^2+2x-3}{x+3} = \lim_{x\to-3}\frac{\cancel{(x+3)}(x-1)}{\cancel{(x+3)}}$

$\displaystyle = \lim_{x\to-3}(x-1) = -4,$

the $\displaystyle\lim_{x\to-3} f(x)$ exists and so f has a removable discontinuity at $x=-3$.

(d) none

(e) $\displaystyle\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{x^2+2x+3}{x+3} = \lim_{x\to\infty}\frac{x^2+2x-3}{x+3}\cdot\frac{1/x}{1/x}$

$\displaystyle = \lim_{x\to\infty}\frac{x+2-3/x}{1+3/x} = \infty$

$\displaystyle\lim_{x\to-\infty} f(x) = -\infty \implies$ no horizontal asymptotes

② (a) $2x^2 + x - 1 = 0$

$(2x - 1)(x + 1) = 0$

$$\boxed{x = 1/2,\ -1}$$

(b) $x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x = -2, 1$

$$\boxed{(-\infty, -2) \cup (-2, 1) \cup (1, \infty)}$$

(c) <u>$x = -2$:</u>

Why? Because $f(-2)$ undefined.

What type?

$$\lim_{x \to -2} \frac{(2x-1)(x+1)}{(x-2)(x-1)} = \text{“}\frac{5}{0}\text{”} \to \text{indicates an } \underline{\text{infinite}} \text{ discont.}$$

<u>$x = 1$:</u>

Why? Because $f(1)$ undefined

What type?

$$\lim_{x \to 1} \frac{(2x-1)(x+1)}{(x-2)(x-1)} = \text{“}\frac{2}{0}\text{”} \to \text{indicates an } \underline{\text{infinite}} \text{ discont.}$$

(d) $-2 \quad -1 \quad 1/2 \quad 1$

−	−	−	+	+	$2x-1$
−	−	+	+	+	$x+1$
−	+	+	+	+	$x+2$
−	−	−	−	+	$x-1$
+	−	+	−	+	$f(x)$

VA at $x = -2$

$$\lim_{x \to -2^-} f(x) = \infty$$

$$\lim_{x \to -2^+} f(x) = -\infty$$

VA at $x = 1$

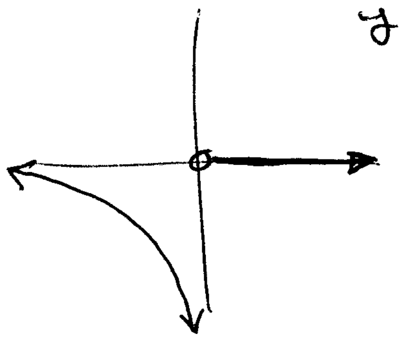
$$\lim_{x \to 1^-} f(x) = -\infty$$

$$\lim_{x \to 1^+} f(x) = \infty$$

(e) $\overleftrightarrow{HA}$: $y = 2$

③ $f(x) = \frac{1}{x} - \frac{1}{|x|} = \begin{cases} 1/x - 1/x & \text{if } x > 0 \\ 1/x - 1/{-x} & \text{if } x < 0 \end{cases} = \begin{cases} 0 & \text{if } x > 0 \\ 2/x & \text{if } x < 0 \end{cases}$

$y = f(x)$



(a) zeros : all x in $(0, \infty)$

(b) $(-\infty, 0) \cup (0, \infty)$

(c) $\underline{x = 0}$:

Why? $f(0)$ undefined

What type? infinite because

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{2}{x} = -\infty$$

(d) VA at $x = 0$

$\lim_{x \to 0^-} f(x) = -\infty$ $\qquad \lim_{x \to 0^+} f(x) = 0$

(e) $\overleftrightarrow{HA}$: $y = 0$

4 (a) $x^4 \cos\left(\frac{2}{x}\right) = 0$

$x^4 = 0$ or $\cos\left(\frac{2}{x}\right) = 0$

$\boxed{x = 0}$ or $\frac{2}{x} = \frac{\pi}{2} + k\pi$, where k is any integer

$\frac{1}{x} = \frac{\pi}{4} + \frac{k\pi}{2}$

$x = \dfrac{1}{\frac{\pi}{4} + \frac{k\pi}{2}} = \dfrac{1}{\frac{\pi + 2k\pi}{4}} = \boxed{\dfrac{4}{\pi + 2k\pi}}$

for example, $k = 0 \Rightarrow \frac{4}{\pi}$

$k = 1 \Rightarrow \frac{4}{3\pi}$ $\quad$ $k = -1 \Rightarrow \frac{4}{-\pi}$

$k = 2 \Rightarrow \frac{4}{5\pi}$ $\quad$ $k = -2 \Rightarrow \frac{4}{-3\pi}$

etc. $\quad$ etc.

(b) Since $2/x$ is a rational ftn, it's cont. on its domain. Since $\cos x$ is a trig function, its continuous on its domain. Since $\cos(2/x)$ is a composition of two cont. ftns, it's cont. on its domain. Since x^4 is a poly. ftn, it's cont. on its domain. Since $x^4\cos\left(\frac{2}{x}\right)$ is a product of two cont. ftns., it's continuous on its domain, $(-\infty, 0) \cup (0, \infty)$.

(c) $\underline{x = 0}$:

Why? Because $f(0)$ undefined.

What type?

$$\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) = ?$$

Since $-1 \leq \cos x \leq 1$

$$\Rightarrow \quad -1 \leq \cos\left(\frac{2}{x}\right) \leq 1$$

$$\Rightarrow \quad -x^4 \leq x^4 \cos\left(\frac{2}{x}\right) \leq x^4$$

Since $\lim_{x \to 0} -x^4 = 0$ and $\lim_{x \to 0} x^4 = 0$,

it follows from the Squeeze Theorem that

$\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) = 0$. Since $\lim_{x \to 0} f(x)$ exists,

$x = 0$ is a <u>removable</u> discont.

(d) none (e) none because

$$\lim_{x \to \infty} x^4 \cos\left(\frac{2}{x}\right) = \infty$$

⑤ (a) $e^x \sin 2x = 0$

$e^x = 0$ or $\sin 2x = 0$

never $\quad 2x = k\pi$ where k is any integer

$$\boxed{x = \frac{k\pi}{2}}$$

for instance, $-\frac{3\pi}{2}$, $-\frac{\pi}{2}$, $\frac{\pi}{2}$, and $\frac{3\pi}{2}$.

(b) $(-\infty, \infty)$

(c) none

(d) none

(e) $\lim\limits_{x \to \infty} e^x \sin 2x = \infty$ since $e^x \to \infty$

and $-1 \le \sin 2x \le 1$

$\lim\limits_{x \to -\infty} e^x \sin 2x = 0$ since $e^x \to 0$

and $-1 \le \sin 2x \le 1$

$\overleftarrow{HA}$ $y = 0$

⑥ (a) $\ln(t^4-1) = 0$

$$e^0 = t^4 - 1$$

$$1 = t^4 - 1$$

$$2 = t^4$$

$$\boxed{t = \pm \sqrt[4]{2}}$$

(b) This function is the composition of two cont. ftns (a logarithm and a polynomial), so it is cont. on its domain.

want $t^4 - 1 > 0$ since domain of $\ln$ is $(0,\infty)$.

$\iff$ $t < -1$ or $t > 1$

$$\boxed{(-\infty, -1) \cup (1, \infty)}$$

(c) Discontinuities at every point of $[-1,1]$ since f is undefined at these points.

Since $\lim\limits_{t \to -1^-} f(x) = \lim\limits_{x \to -1^-} \ln(t^4-1) = \lim\limits_{u \to 0^+} \ln(u) = -\infty$

and $\lim\limits_{t \to 1^+} f(x) = \lim\limits_{t \to 1^+} \ln(t^4-1) = \lim\limits_{u \to 0^+} \ln(u) = -\infty$

both $t = -1$ and $t = 1$ are <u>infinite</u> discontinuities.

(d) VA at $t=-1$

$$\lim_{t\to -1^-} f(x) = -\infty$$

VA at $t=1$

$$\lim_{t\to 1^+} f(t) = -\infty$$

(e) $$\lim_{t\to\infty} \ln(t^4-1) = \lim_{u\to\infty} \ln u = \infty$$

$$\lim_{t\to-\infty} \ln(t^4-1) = \lim_{u\to\infty} \ln u = \infty$$