Solutions to the First Examination

Mathematics 1a: Fall, 2003

1. (a) Find $f'(x)$ if $f(x) = 3^{x^2} + \dfrac{1}{x \ln \pi} + \sin^2(3x)$. You need not simplify.

Applying the chain rule to each term in the sum we have,

$$\frac{df}{dx} = (3^{x^2} \ln 3)(2x) + \left(\frac{-1}{(x \ln \pi)^2}\right)(\ln \pi) + (2 \sin 3x)(\cos 3x)(3)$$

(b) Let $f(x) = \arctan(4x) \cdot \ln[\tan(\pi x)]$.

i. Find $f'(x)$ You need not simplify.

Applying the product rule and the chain rule,

$$f'(x) = \left(\frac{4}{1+(4x)^2}\right)(\ln[\tan(\pi x)]) + (\arctan(4x))\left(\frac{\pi \sec^2 \pi x}{\tan \pi x}\right)$$

ii. Find $f'(1/4)$ Simplify as much as possible.

Since $\arctan x$ is the inverse function of tangent and $\sec x = 1/\cos x$, we have $\arctan 1 = \pi/4$ and $\sec(\pi/4) = 1/(1/\sqrt{2}) = \sqrt{2}$. So,

$$\begin{aligned} f'(1/4) &= \frac{4 \ln[\tan(\pi/4)]}{1+1^2} + \frac{(\arctan 1)(\pi)(\sec^2 \pi)}{\tan \pi/4} \\ &= \frac{4 \ln 1}{2} + \frac{(\pi/4)(\pi)(2)}{1} \\ &= 0 + \frac{\pi^2}{2} \\ &= \frac{\pi^2}{2} \end{aligned}$$

(c) If $f(x) = \dfrac{x^2(x^{1/2})^3 + 8x}{2\sqrt{x}}$, then $f'(4) =$

(i) 0 (ii) 52/16 (iii) 23/16 (iv) 25
(v) 3 (vi) 2 (vii) 52/8 (viii) 32
(ix) none of the above

We collect all powers of x to obtain $f(x) = x^3/2 + 4x^{1/2}$. Differentiating, $f'(x) = 3x^2/2 + 2x^{-1/2}$. Substituting $x = 4$, $f'(4) = 25$, which is answer (iv).

(d) Find $y'(x)$ if $y = \dfrac{4x^{3x+1}}{7}$.

There are two ways to approach this problem.

Method 1: We use logarithmic differentiation. Thus,

$$\begin{aligned} \ln y &= \ln\left(\frac{4}{7} x^{3x+1}\right) \\ &= \ln(4/7) + \ln x^{3x+1} \\ &= \ln(4/7) + (3x+1)\ln x, \end{aligned}$$

and so, differentiating implicitly,

$$\frac{1}{y} y' = 3 \ln x + \frac{3x+1}{x}.$$

Consequently,

$$y' = y\left(3 \ln x + \frac{3x+1}{x}\right) = \frac{4x^{3x+1}}{7}\left(3 \ln x + \frac{3x+1}{x}\right).$$

Method 2: We rewrite the function with an exponential:

$$y = \frac{4}{7} e^{\ln(x^{3x+1})} = \frac{4}{7} e^{(3x+1)\ln x},$$

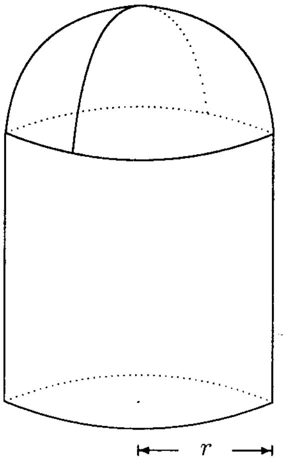
and differentiating,

$$y' = \frac{4}{7} e^{(3x+1)\ln x} \left(3\ln x + \frac{3x+1}{x}\right) = \frac{4x^{3x+1}}{7} \left(3\ln x + \frac{3x+1}{x}\right).$$

(e) Consider the graph of the equation $y^2 + xy = x^3 + 1$. Compute the slope of the tangent line to this curve at the point $(-1, 1)$.
We use implicity differentiation, remembering the product rule:

$$\begin{aligned} 2yy' + y + xy' &= 3x^2 \\ (2y + x)y' &= 3x^2 - y \\ y' &= \frac{3x^2 - y}{2y + x}. \end{aligned}$$

Substituting $x = -1$ and $y = 1$, we find that $y' = 2$ at the point $(-1, 1)$.



2.

Let r be the radius and h be the height of the cylinder. We want to minimize the cost:

$$\begin{aligned} C &= \text{(cost of the roof)} + \text{(cost of the wall)} + \text{(cost of the base)} \\ &= 7 \cdot (\frac{1}{2} \cdot 4\pi r^2) + 3 \cdot (2\pi r h) + 3 \cdot (\pi r^2) \\ &= 17\pi r^2 + 6\pi r h, \end{aligned}$$

under the conditions:

$$\begin{aligned} 100 = \text{total volume} &= \text{(volume of the hemisphere)} + \text{(volume of the cylinder)} \\ &= \frac{1}{2} \cdot \frac{4\pi r^3}{3} + \pi r^2 h \\ &= \frac{2\pi r^3}{3} + \pi r^2 h \qquad (*) \end{aligned}$$

and $r > 0$; $h > 0$.
¿From $(*)$,

$$h = \frac{100 - \frac{2}{3}\pi r^3}{\pi r^2} = \frac{100}{\pi r^2} - \frac{2r}{3}.$$

Therefore

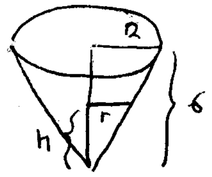
$$\begin{aligned} C &= 17\pi r^2 + 6\pi r \cdot (\frac{100}{\pi r^2} - \frac{2r}{3}) \\ &= 13\pi r^2 + \frac{600}{r}. \end{aligned}$$

which is now a function of one variable r. The condition $h > 0$ implies $100 - \frac{2}{3}\pi r^3 > 0$ i.e.

$$r < \sqrt[3]{\frac{150}{\pi}}.$$

Thus the function we want to minimize is $13\pi r^2 + \frac{600}{r}$ and the domain of the function is $0 < r < \sqrt[3]{\frac{150}{\pi}}$.

3. The volume of the cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$. Both the radius and the height of the part filled with water increase, and we can relate them using similar triangles: $r/2 = h/6$, so $r = h/3$ (at any time). Then,



$$V = \frac{1}{3}\pi(h/3)^2 h = \frac{1}{3}\pi\frac{h^3}{27} = \frac{\pi}{27}h^3,$$
$$\frac{dV}{dt} = \frac{\pi}{27}h^2\frac{dh}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}.$$

When water is 1 inch from top, $h = 5$ and $\frac{dV}{dt} = 9$, so $9 = \frac{\pi}{9}25\frac{dh}{dt}$, hence $\frac{dh}{dt} = \frac{81}{25\pi}$.

4. **Solution:** The volume of the box is V = width × length × height. In our case $V = xy(30 - 2x)$. Since $2x + 2y = 30$ (length of the side of the square), $x = 15 - y$ and $30 - 2x = 30 - (30 - 2y) = 2y$. Hence

$$V = 2y^2(15 - y);$$

$$V' = 4y(15 - y) - 2y^2 = y(60 - 4y - 2y) = 6y(10 - y).$$

So, $V(y)$ is continuous function on a closed interval $0 \le y \le 15$ with critical points at $y = 0$ and $y = 10$. Values of V at critical points and at ends of the interval are:

$V(0) = 0, \quad V(10) = 2 \cdot 10^2 \cdot 5 = 1000, \quad V(15) = 0.$

By Closed Interval Method $V(y)$ has global maximum at $y = 10$.

Answer: $V_{max} = 1000$ cm^3.

5. (a) We know that any continuous function f on a closed interval $[a, b]$ has a global maximum and a global minimum, so clearly the function f that we choose will not be continuous.

One possible example is the function f defined over $[0, 1]$ by $f(x) = 0$ for $0 < x \le 1/2$ and $f(x) = -x + 1$ for $1/2 < x \le 1$.

(b) This is false. If we take $f(x) = x^3$ then $f'(x) = 3x^2$ and $f'(0) = 0$. However, f clearly has neither a maximum nor a minimum at $x = 0$.

(c) If f is differentiable and $df/dx = x^2 - 1$, then the critical points of f are $x = 1$ and $x = -1$. Using the first derivative test, we see that df/dx changes sign from positive to negative so f has a local maximum at $x = -1$. Similarly, at $x = 1$ df/dx changes sign from negative to positive, so f has a local minimum at $x = 1$.

(d) If $f'(2) = 0$ then we know that 2 is a critical point of f. We use the second derivative test: Since $f''(2) = 1 > 0$ it follows that $x = 2$ is a local minimum for f.

(e) $lim_{n\to\infty} M_0(1 + .04/n)^{nt} = M_0 e^{.04t}$

6. (a) $x : x < 0$, i.e. $(0, \infty)$

Reason: x is defined for all numbers, but $\ln x$ is defined only for positive x.

(b) Notice that the limit is of the form $0 \cdot \infty$, so we must put it into the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$ before applying L'Hospital's Rule.

$$\begin{aligned}
\lim_{x\to 0^+} x(\ln x)^2 &= \lim_{x\to 0^+} \frac{(\ln x)^2}{1/x} \quad (\frac{\infty}{\infty} \text{ apply L'Hospital's Rule)} \\
&= \lim_{x\to 0^+} \frac{(2\ln x)(1/x)}{-1/x^2} \\
&= \lim_{x\to 0^+} \frac{2\ln x}{-1/x} \quad (\frac{\infty}{\infty} \text{ apply L'Hospital's Rule)} \\
&= \lim_{x\to 0^+} \frac{(2/x)}{1/x^2} \\
&= \lim_{x\to 0^+} 2x = 0
\end{aligned}$$

(c) Notice that we don't have an indeterminate form here. L'Hospital's Rule does not apply. $\lim_{x\to\infty} x(\ln x)^2$ is of the form $\infty \cdot \infty$. As x grows without bound, so do both x and $(\ln x)^2$. Therefore, $\lim_{x\to\infty} x(\ln x)^2 = \infty$.

(d) It may be helpful to first notice that $f(x) = x(\ln x)^2$ is never negative, since the domain is all positive x and anything squared is non-negative. $f(x) = x(\ln x)^2 = 0$ only when $\ln x = 0$, that is, only at $x = 1$. This information indicates to us that there is an absolute minimum at $x = 1$ and putting this together with (b) we know that we should expect another critical point (this one a local maximum) between 0 and 1.

Critical points are points in the domain of f at which $'f$ is undefined or zero.
$f'(x) = (\ln x)^2 + x \cdot 2\ln x \cdot (1/x) = (\ln x)^2 + 2\ln x$

$$\begin{aligned}
(\ln x)^2 + 2\ln x &= 0 \\
(\ln x)(\ln x + 2) &= 0
\end{aligned}$$

$\ln x = 0$ or $\ln \mathrm{x} + 2 = 0$, equivalently, $\ln x = 0$ or $\ln \mathrm{x} = -2$
If $\ln x = 0$ then $x = e^0 = 1$. If $\ln x = -2$, then $x = e^{-2}$.

You can use the first or second derivative test to classify these two critical points.

First derivative test:
Around e^{-2} the sign of f' changes from positive to negative and around 1 the sign of f' changes from negative to positive, making $x = e^{-2}$ a local maximum and $x = 1$ a local minimum.

Second derivative test: $f''(x) = 2\ln x(1/x) + (2/x) = \frac{2(\ln x+1)}{x}$.
At $x = e^{-2}$, we have $f''(e^{-2}) = \frac{2(\ln(e^{-2})+1}{e^{-2}} = 2e^2(-2+1) < 0$ implies f is concave down at $x = e^{-2}$ and therefore has a local maximum at $x = e^{-2}$.
$f(e^{-2}) = e^{-2}(\ln(e^{-2})^2 = e^{-2}(4) = 4e^{-2}$, so the point is $((e^{-2}, 4e^{-2})$.

At $x = 1$, we have $f''(1) = \frac{2(\ln(1)+1}{1} = 2(1+1) > 0 \implies f$ is concave up at $x = 1$ and therefore has a local minimum at $x = 1$.
The point is $(1, 0)$.

(e) f has an absolute minimum value of 0 when $x = 1$. For every other x in the domain, f is positive.

(f) f is concave down where $f'' < 0$.
$f''(x) = 2\ln x(1/x) + (2/x) = \frac{2(\ln x+1)}{x}$
$f''(x)$ can change sign across points where it is zero or undefined. f'' is never undefined on the domain of f.
$f''(x) = 0$ where $\ln x = -1$, that is, where $x = e^{-1}$.
looking at the sign of $f''(x)$, we see that it is negative for $(0, e^{-1})$.