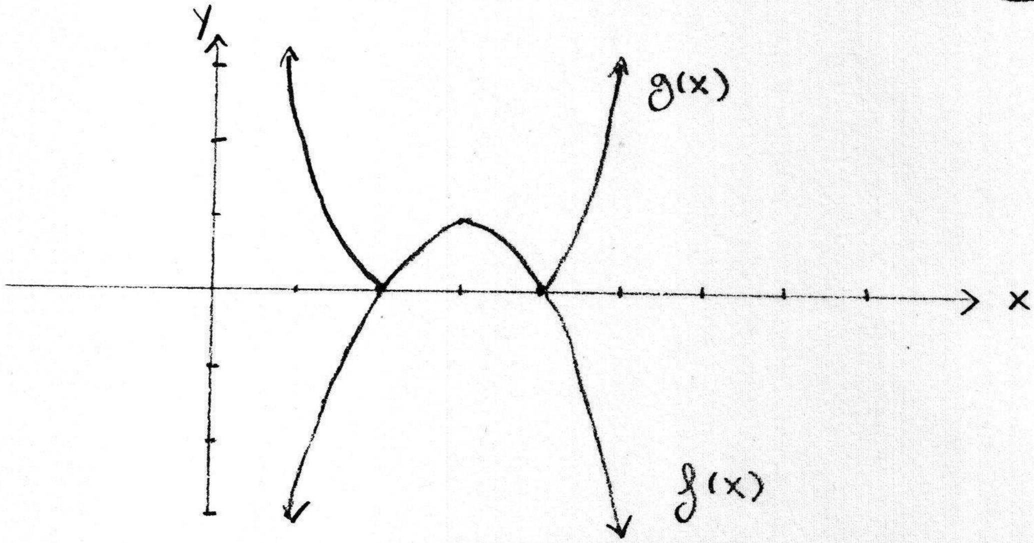
Problem Sheet C solutions

BY EDUARDO SAVERIN

Question 1

a) $f(x) = (x-2)(-x+4) = -x^2 + 2x + 4x - 8 = \boxed{-x^2 + 6x - 8}$

$g(x) = |f(x)| = |(x-2)(-x+4)| = \boxed{|-x^2 + 6x - 8|}$




b) Consider the function: $f(x) = -x^2 + 6x - 8$

It follows that ...

$f'(x) = -2x + 6 \qquad (1)$

$f''(x) = -2 \qquad (2)$

Derivation of (1) ...

① $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$\Rightarrow \lim_{h \to 0} \dfrac{-(x+h)^2 + 6(x+h) - 8 + x^2 - 6x + 8}{h}$

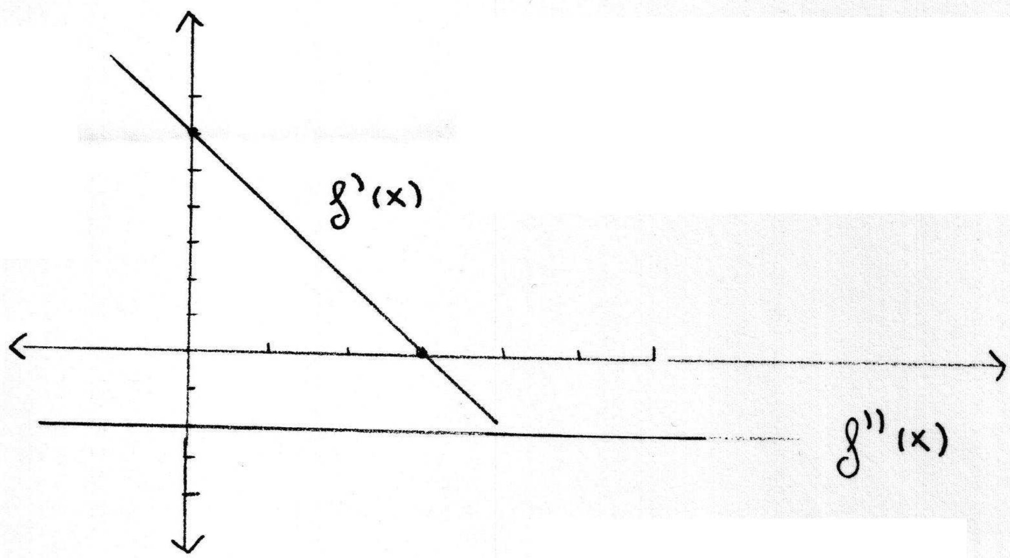
$$\Rightarrow \lim_{h \to 0} \frac{-h^2 - 2xh + 6h}{h}$$

$$\Rightarrow \lim_{h \to 0} -h + 6 - 2x = \boxed{-2x + 6}$$

② $$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(h)}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{-2(x+h) + 6 + 2x - 6}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{-2h}{h} = \boxed{-2}$$




c) Refer to part a)

d) Consider the function: $g(x) = |-x^2 + 6x - 8|$

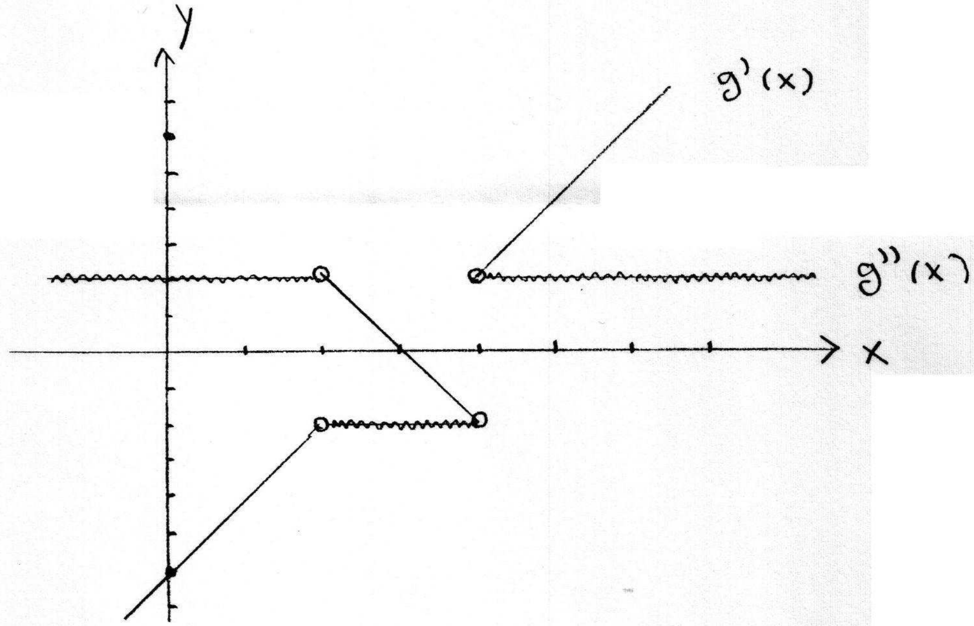
This function can be represented piecewise by...

$$g(x) = \begin{cases} x^2 - 6x + 8, & x < 2 \\ -x^2 + 6x - 8, & 2 \leq x \leq 4 \\ x^2 - 6x + 8, & x > 4 \end{cases}$$

Given the results to part b), it follows that ...

$$g'(x) = \begin{cases} 2x-6 & , \ x<2 \\ -2x+6 & , \ 2<x<4 \\ 2x-6 & , \ x>4 \end{cases}$$

$$g''(x) = \begin{cases} 2 & , \ x<2 \\ -2 & , \ 2<x<4 \\ 2 & , \ x>4 \end{cases}$$




e) The function $g(x)$ has two corner points, at $x = 2$ and at $x = 4$. Neither the first ($g'(x)$) nor the second ($g''(x)$) derivative of $g(x)$ are defined at these points - this fact is supported by the graph in part d).

$g'(x)$ AND $g''(x)$ are undefined at $x = 2, 4$

f) Consider the piecewise definition of $g'(x)$ ...

① $\lim_{x \to 4^+} g'(x) \Rightarrow \lim_{x \to 4^+} 2x - 6 = 2(4) - 6 = \boxed{2}$

② $\lim_{x \to 4^-} g'(x) \Rightarrow \lim_{x \to 4^-} -2x + 6 = -2(4) + 6 = \boxed{-2}$

g) Consider the piecewise definition of $g'(x)$ ...

① $\lim_{x \to 2^+} g''(x) = \boxed{-2}$

② $\lim_{x \to 2^-} g''(x) = \boxed{2}$

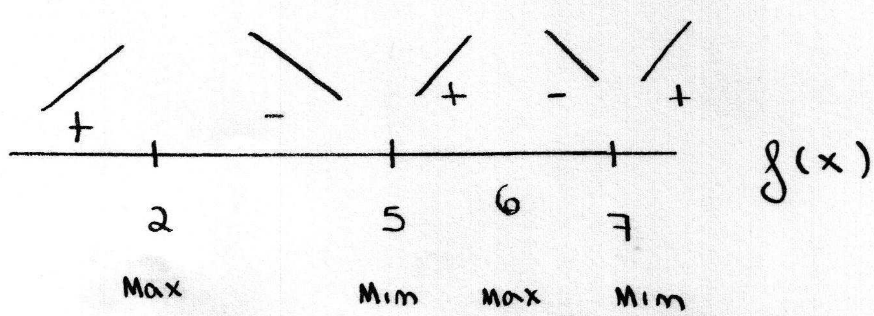
## Question 2

a, b) The function $f(x)$ can only have local maxima and minima in stationary points, where $f'(x) = 0$.

Stationary Points of $f(x)$ : $\boxed{x = 2, 5, 7}$ Note: $(6, f(6))$ is a corner point

Now, determine whether the given stationary points are local maxima or minima ...

+ : increasing
- : decreasing

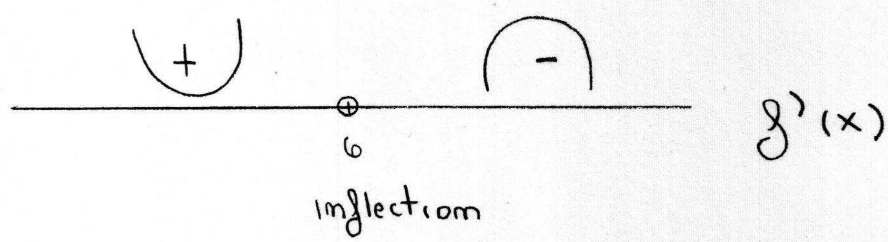


It follows that:

| |
|---|
| $f(x)$ Local maxima: $x = 2$ , $x = 6$ |
| $f(x)$ Local minima: $x = 5$ , $x = 7$ |

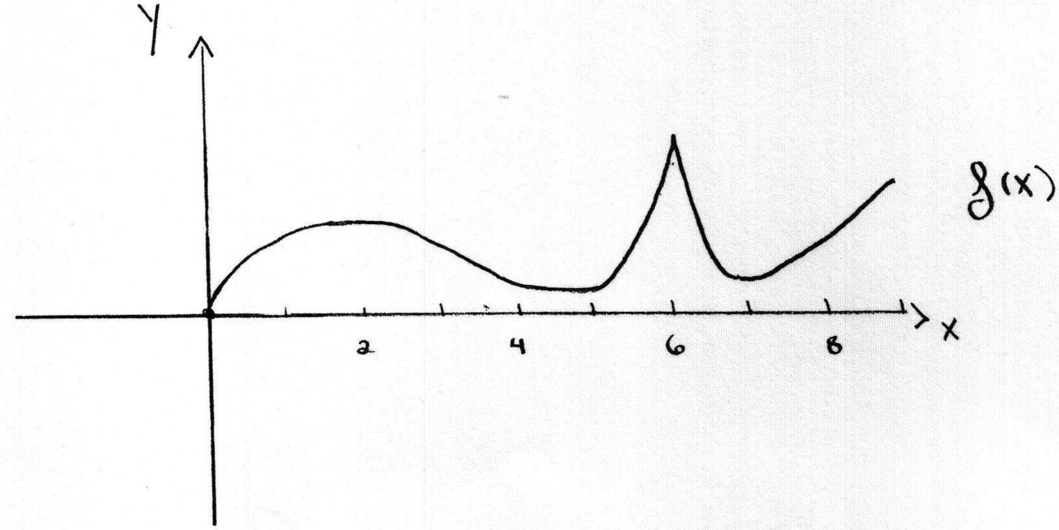
c) $f(x)$ is concave up ($f''(x) > 0$) for $x \in [0, 6)$

$f(x)$ is concave down ($f''(x) < 0$) for $x \in (6, 9]$




It follows that $f(x)$ has a point of inflection at $x = 6$.

d) A possible graph of $f(x)$ follows...



## Question 3

Let $f(x)$ be a function whose derivative is $f'(x) = |x|$

Note that $f'(x) = |x|$ can be represented in piecewise function as...

$$f'(x) = \begin{cases} x & , x > 0 \\ -x & , x < 0 \end{cases}$$

It follows that $f(x)$ can be a function of the form...

$$\boxed{f(x) = \begin{cases} \frac{x^2}{2} & , x > 0 \\ \frac{-x^2}{2} & , x < 0 \end{cases}}$$

from the family of functions of parabolas

## Question 4

a) $a(t)$, the creature's acceleration function, is the derivative of $v(t)$, the creature's velocity function.

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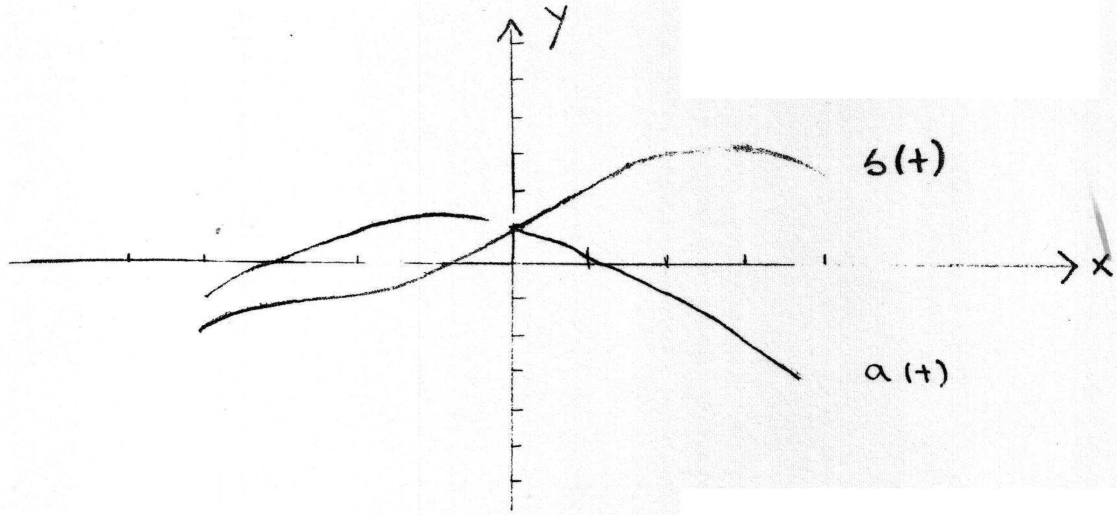


b) Refer to the graph above