Math 1, 2002 Midterm

Problems # 1, 2 by Peter Zoogman

1a) $\frac{d}{dx}(x^3 \sec x \ln x) = 3x^2 \sec x \ln x + x^3 \sec x \tan x \ln x + x^3 \sec x \cdot \frac{1}{x}$

$= x^2 \sec x (3 \ln x + x \tan x \ln x + 1)$

b) $\frac{d}{dx} \sin e^{x^2} = \cos e^{x^2} \cdot \frac{d}{dx} e^{x^2}$

$= \cos e^{x^2} \cdot 2x e^{x^2}$

c) $\frac{d}{dx}\left(x \log_3 x + \frac{3^x}{x}\right) = \log_3 x + x \cdot \frac{1}{x \ln 3} + \frac{x 3^x \ln 3 - 3^x}{x^2}$

$= \log_3 x + \frac{1}{\ln 3} + \frac{3^x(x \ln 3 - 1)}{x^2}$

d) $\frac{d}{dx}\left(\tan^{-1}(3x^2)\right) = \frac{1}{1+(3x^2)^2} \cdot 6x = \frac{6x}{1+9x^4}$

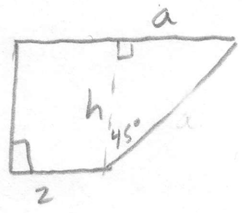
e) $\frac{d}{dx}\left(4(\tan x)^5\right) = 4 \frac{d}{dx} \tan^5 x = 20 \tan^4 x \cdot \sec^2 x$

f) $\frac{d}{dx}\left(\cos(x + \cos x)\right) = -\sin(x + \cos x) \cdot (1 - \sin x)$

g) $(2+\cos x)^x = y \Rightarrow x \ln(2+\cos x) = \ln y$

$\ln(2+\cos x) + \frac{x}{2+\cos x} \cdot (-\sin x) = \frac{1}{y} y'$

$\frac{dy}{dx} = (2+\cos x)^x \left[\ln(2+\cos x) - \frac{x \sin x}{2+\cos x}\right]$

2a) 

$a = h$

For area, we can add the areas of the square and the triangle to find the area of the trapezoid:

$$A(h) = 2h + \tfrac{1}{2}h \cdot h$$
$$= 2h + \tfrac{1}{2}h^2$$

b) given $\frac{dA}{dt} = -1 \text{ ft}^2/\text{min}$

by chain rule $\frac{dA}{dt} = \frac{dA}{dh} \cdot \frac{dh}{dt}$ (1)

and from part (a) we see $\frac{dA}{dh} = 2 + h$ (2)

so $\frac{dh}{dt} = \frac{-1}{2+h}$

and at $h=2$, $\frac{dh}{dt} = \frac{-1}{4}$ ft/min

c) given $\frac{dh}{dt} = 1$ ft/min

we still know (1) and (2) from above, so we see $\frac{dA}{dt} = (2+h)(1)$

So when $h=2$, $\frac{dA}{dt} = 4 \text{ ft}^2/\text{min}$

3. $f(x) = x^3 + 3x^2 - 24x + 8$

a) $f'(x) = 3x^2 + 6x - 24$

$3x^2 + 6x - 24 = 0 \iff x^2 + 2x - 8 = 0 \Rightarrow x = -4$
or $x = 2$

$\Rightarrow f'(x)$: + (−4) − (2) +

$\Rightarrow f$ decreasing on $(-4, 2)$, increasing on $(-\infty, -4)$; $(2, +\infty)$

b) local max : $x = -4 \Rightarrow f = 88$
local min : $x = 2 \Rightarrow f = -20$

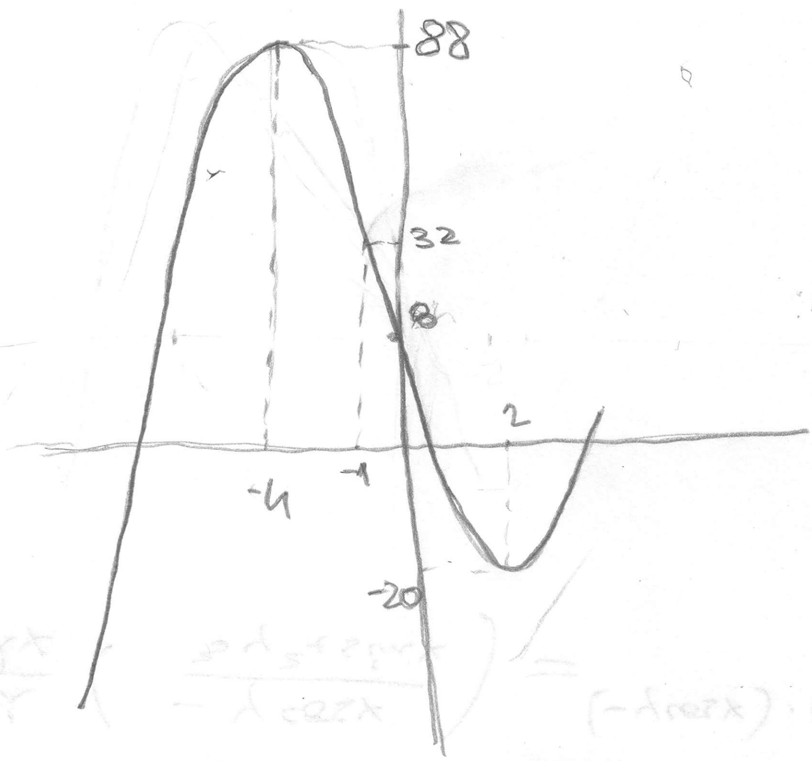
c) $f''(x) = 6x + 6 \Rightarrow 6x + 6 = 0 \iff x = -1$

$\Rightarrow f''(x)$: − (−1) +

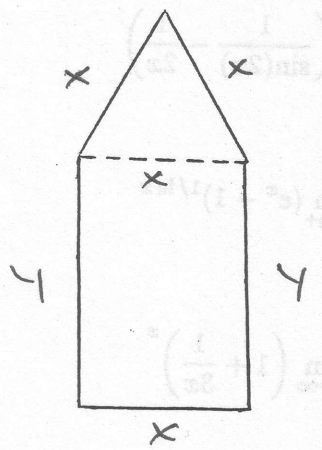
$\Rightarrow$ concave up on $(-1, +\infty)$
concave down on $(-\infty, -1)$

d) $x = -1 \Rightarrow$ inflection point

e)



4. (14 points) A window with the shape of a rectangle capped by an equilateral triangle (see the figure) is to have a perimeter of 100 feet. What is the largest possible area of such a window?



$$\Rightarrow 3x + 2y = 100 \quad \Rightarrow y = \frac{100-3x}{2}$$

$$A = xy + \frac{x^2\sqrt{3}}{4} = x \cdot \frac{100-3x}{2} + \frac{x^2\sqrt{3}}{4} = \frac{200x - 6x^2 + \sqrt{3}}{4}$$

$$\frac{dA}{dx} = \frac{200 - 12x + 2\sqrt{3}x}{4} = 0$$

$$\Rightarrow 200 = 12x - 2\sqrt{3}x \Rightarrow x = \frac{200}{12-2\sqrt{3}} = \frac{50(12+2\sqrt{3})}{33}$$

$$\Rightarrow y = \frac{100 - \frac{50(12+2\sqrt{3})}{11}}{2} = \frac{25}{11}(10-2\sqrt{3})$$

$$\Rightarrow A_{max} = \frac{2500\sqrt{3} + 15000}{33}$$

5)

(A) $\lim_{x \to \pi/2} \frac{\sin x}{x} = \frac{\sin \pi/2}{\pi/2} = \frac{1}{\pi/2} = \boxed{\frac{2}{\pi}}$

(B)

$\lim_{x \to 0} \left(\frac{1}{\sin(2x)} - \frac{1}{2x} \right)$ = let $x = n$

$\lim_{n \to 0} \left(\frac{1}{\sin} - \frac{1}{n} \right) = \lim_{n \to 0} \frac{n - \sin(n)}{n \sin(n)}$

By l'Hopital's rule

$= \lim_{n \to 0} \frac{1 - \cos n}{n(\cos(n)) + \sin n}$ $\rightarrow$ limit does not exist

(C) $\lim_{x \to 0^+} (e^x - 1)^{1/\ln x} \rightsquigarrow e^{\lim_{x \to 0^+} \ln (e^x - 1)^{1/\ln x}}$

$\lim_{x \to 0^+} \frac{\ln(e^x - 1)}{\ln x} = \lim_{x \to 0^+} \frac{x}{(e^x - 1)e^x} = 1$

$\lim_{x \to 0^+} (e^x - 1)^{1/\ln x} = e^1$

(D) $\lim_{x \to \infty} \left(\frac{1}{1 + 3x} \right)^x$ let $n = 3x$ $x = \frac{n}{3}$

$\lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^{n/3} = \lim_{n \to \infty} \left(\left(1 + \frac{1}{n} \right)^n \right)^{1/3}$

$= e^{1/3}$ $\hookrightarrow$ $\left(\lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n \right)^{1/3}$

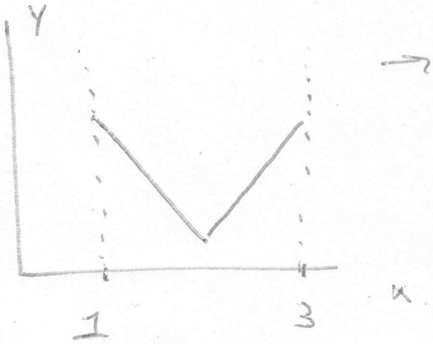
Nov. 26 '02

#6 A If $f(x)$ is differentiable on $[a, b]$

then there exists a "c" such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$ where $a \le c \le b$

B



→ continuous but not differentiable

Given:

C $f(-2) = 1$ and $f'(x) < 3$

show $f(2) < 13$

let $b = 2$ and $a = -2$

From M.V.T. $f(b) - f(a) = f'(c)(b-a)$

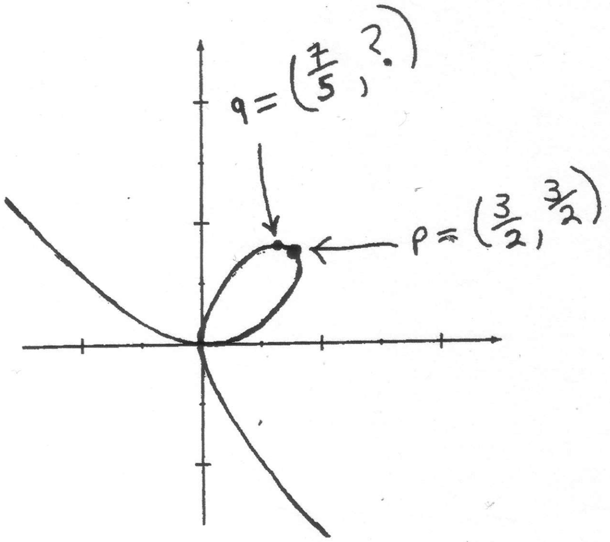
$$f(b) = f'(c)(b-a) + f(a)$$

$$f(2) = f'(c)(4) + 1$$ → from above $f'(c) < 3$

so $f'(c)(4) + 1 < 13$

Nov. 26 '02 $f(2) < 13$

7. Consider the graph of the equation $x^3 + y^3 = 3xy$, as shown in the figure below:



(a) (4 points) Find $\frac{dy}{dx}$ in terms of x and y.

(b) (4 points) Find the equation of the line tangent to the graph at the point $p = (\frac{3}{2}, \frac{3}{2})$.

(c) (4 points) Use the tangent line in part (b) to approximate the y-coordinate of the point labeled q, using the fact that the x-coordinate of q is $\frac{7}{5}$.

a) $x^3 + y^3 = 3xy$

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(3xy)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 3x^2$$

$$\frac{dy}{dx}(3y^2 - 3x) = 3y - 3x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

b) $m = \frac{y - x^2}{y^2 - x} = \frac{\frac{3}{2} - \frac{9}{4}}{\frac{9}{4} - \frac{3}{2}} = \frac{-\frac{3}{4}}{\frac{3}{4}} = -1$

$$y - \frac{3}{2} = -1(x - \frac{3}{2})$$

$$y = -x + 3$$

c) Use a linearization around $x = \frac{3}{2}$ to estimate y at $x = \frac{7}{5}$

$L(x) = f(a) + f'(a)(x - a)$ $a = \frac{3}{2}$

$$= \frac{3}{2} + (-1)(\frac{7}{5} - \frac{3}{2})$$

$$= \frac{3}{2} - \left(\frac{14 - 15}{10}\right) =$$

$$= \frac{15}{10} + \frac{1}{10} = \frac{16}{10} = \frac{8}{5}$$

8.

$f(x) = \frac{\sin x}{x}$

$f(x) = 0$ at $n\pi$ where n is an integer. (D or E)

$f(x)$ is sinsodial (C, D, or E)

$\lim_{x \to 0} \frac{\sin x}{x} = 1$ (D)

These correspond to D.

$g(x) = \frac{1-x}{e^x}$

$g'(x) = \frac{e^x(-1) - e^x(1-x)}{e^{2x}}$

$= \frac{(-2+x)e^x}{(e^x)^2} = \frac{x-2}{e^x}$

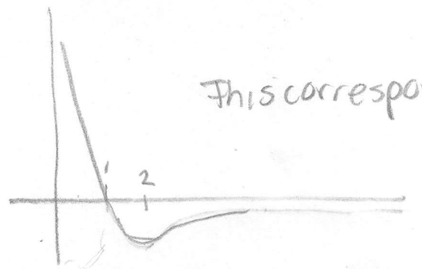
Note: since $e^x > 0$ for all x

$g(x) > 0$ if $x < 1$

$g'(x) > 0$ if $x > 2$

$g''(x) = \frac{e^x(1) - e^x(x-2)}{(e^x)^2} = \frac{3-x}{e^x}$

$g''(x) > 0$ if $x < 3$



This corresponds to B.

$h(x) = -\frac{\ln x}{x}$ $\quad h < 0$ if $x > 1$

$h'(x) = \frac{x(-\frac{1}{x}) + \ln x}{x^2} = \frac{-1 + \ln x}{x}$

$h'(x) = 0$ if $x = e$

$h'(x) > 0$ if $x > e$

This corresponds to A

$h(x) = \frac{\cos x}{x}$

$h'(x) = \frac{-\sin x (x) - \cos x}{x^2} = \frac{-\sin x}{x} - \frac{\cos x}{x^2}$

$h''(x) = \frac{-x\cos x + \sin x}{x^4} + \frac{x^2 \sin x + \cos x (2x)}{x^4}$

$= \frac{x\cos x + \sin x + x^2 \sin x}{x^4}$

$\left.\begin{array}{l} h(x) < 0 \text{ at } \pi, 3\pi, 5\pi \ldots \\ < 0 \text{ at } 2\pi, 4\pi \ldots \\ = 0 \text{ at } \pi/2, 3\pi \end{array}\right\}$ just some easy pts to look for

and, since $h(x)$ is sinsodial (C, D, or E)

the only possibility is C.