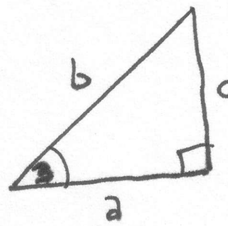
2000 Midterm

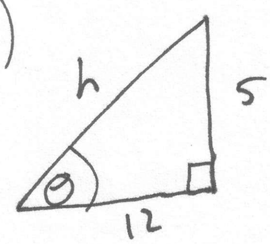
1. a)
$$\frac{\log_5 16 \cdot \log_2 9}{\log_5 3} = \frac{\log_5(2^4)\cdot \log_2(3^2)}{\log_5 3}$$
$$= \frac{8 \log_5 2 \cdot \log_2 3}{\log_5 3} = 8 \log_3 2 \cdot \log_2 3 = 8$$

b)



$$\cos 3 = \frac{a}{b}$$
$$\sin^{-1}\left(\frac{a}{b}\right) = \boxed{\frac{\pi}{2} - 3 = \arcsin(\cos(3))}$$

c)



$$h = \sqrt{5^2 + 12^2} = 13$$
$$\theta = \arctan\left(\frac{5}{12}\right)$$
$$\cos\theta = \frac{12}{13}$$
$$\boxed{\cos\left(\arctan\left(\frac{5}{12}\right)\right) = \frac{12}{13}}$$

2.

a) $\frac{d}{dx}\left(\ln \arccos(x)\right) = \frac{1}{\arccos(x)} \cdot \left(\arccos(x)\right)' =$

$= \frac{1}{\arccos(x)} \cdot \left(-\frac{1}{\sqrt{1-x^2}}\right) = \boxed{-\frac{1}{\sqrt{1-x^2} \cdot \arccos(x)}}$

b) $\frac{d}{dx}\left(\arctan(x)\right)^x$

let $(\arctan(x))^x = y \quad \left(\Rightarrow y' = \frac{d}{dx}(\arctan(x))^x \right)$

$\Rightarrow \ln\left[(\arctan(x))^x\right] = \ln y$

$\Rightarrow x \cdot \ln(\arctan(x)) = \ln y$

$\Rightarrow \ln(\arctan(x)) + x \cdot \frac{(\arctan(x))'}{\arctan x} = \frac{y'}{y}$

$\Rightarrow \boxed{y' = \left[\ln(\arctan(x)) + \frac{x}{(1+x^2)\cdot \arctan x}\right] \cdot (\arctan(x))^x}$

c) $y^3 + y \cdot \sin x - 1 = 0$

$\Rightarrow 3y^2 \cdot \frac{dy}{dx} + \frac{dy}{dx} \cdot \sin x + y \cdot \cos x = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-y \cdot \cos x}{3y^2 + \sin x}$

at $x = 0 \Rightarrow y^3 - 1 = 0 \Rightarrow y = 1$

$\Rightarrow \left.\frac{dy}{dx}\right|_{\text{at}(0,1)} = \frac{-1 \cdot 1}{3 \cdot 1 + 0} = \boxed{-\frac{1}{3}}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{-y\cos x}{3y^2+\sin x}\right) =$$

$$= \frac{(-y\cos x)\cdot\left(6y\cdot\frac{dy}{dx}+\cos x\right) - \left(-\frac{dy}{dx}\cos x + y\sin x\right)(3y^2+}{(3y^2+\sin x)^2}$$

$$\underset{\text{at}(0,1)}{=} \frac{(-1)(-1)-\left(\frac{1}{3}\right)\cdot 3}{9} = \boxed{0}$$

(3)

$$A = \frac{1}{2} ab \sin x$$

$$\frac{dA}{dt} = \frac{1}{2} b \sin x \frac{da}{dt} + \frac{1}{2} a \sin x \frac{db}{dt} + \frac{1}{2} ab \cos x \frac{dx}{dt}$$

since $A = \frac{1}{2}$ sq in $=$ constant $\frac{dA}{dt} = 0$

$$\therefore \quad \frac{dx}{dt} = \frac{b \sin x \frac{da}{dt} + a \sin x \frac{db}{dt}}{ab \cos x}$$

What is x? We know $A = \frac{1}{2} ab \sin x$, $A = \frac{1}{2}$, $a = 1$, $b = \sqrt{2}$

$$\therefore \sin x = \frac{\frac{1}{2}}{\frac{1}{2} \cdot 1 \cdot \sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\therefore x = \frac{\pi}{4} \quad (\because x \text{ is acute})$$

hence $\sin x = \frac{1}{\sqrt{2}}$, $\cos x = \frac{1}{\sqrt{2}}$

$$\therefore \quad \frac{dx}{dt} = \frac{\sqrt{2} \cdot \frac{1}{\sqrt{2}} \frac{da}{dt} + 1 \cdot \frac{1}{\sqrt{2}} \frac{db}{dt}}{1 \cdot \sqrt{2} \cdot \frac{1}{\sqrt{2}}}$$

$$\frac{dx}{dt} = \frac{da}{dt} + \frac{1}{\sqrt{2}} \frac{db}{dt}$$

Since we are not told whether a and b are increasing or not,

$$\frac{da}{dt} = \pm 1, \quad \frac{db}{dt} = \pm 2$$

$\therefore \frac{dx}{dt} = 1 + \frac{2}{\sqrt{2}}$ or $1 - \frac{2}{\sqrt{2}}$ or $-1 + \frac{2}{\sqrt{2}}$ or $-1 - \frac{2}{\sqrt{2}}$ radians/second.

$(a\nearrow, b\nearrow)$ $(a\nearrow, b\downarrow)$ $(a\downarrow, b\nearrow)$ $(a\downarrow, b\downarrow)$

#4 (A) $\lim_{x \to 0} \frac{\sin x - x}{x^3} = \frac{0}{0}$ $\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{f(x)}{g(x)}$

L'Hopital's

$= \frac{\cos x}{3x^2}$

$= \frac{-\sin x}{6x}$

$= \frac{-\cos x}{6}$ $\rightarrow$ no you can plug in.

$\boxed{= \frac{-1}{6}}$

(B)

$\lim_{x \to \infty} (1+x)^{1/x}$ let $x = \frac{1}{n}$ so as $x \to \infty$, $n \to 0$

$= \lim_{n \to 0} \left(1 + \frac{1}{n}\right)^n = (1)^0 = \boxed{1}$

(C)

$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \boxed{e}$ by definition of e.

(D)

$\lim_{x \to 2} \frac{3^x - 9}{x-2} = \frac{0}{0}$ $\frac{d}{dx} a^x = a^x \ln a$

$= \frac{3^x \ln 3}{1} = \boxed{9 \ln(3)}$

(E) $\rightarrow$ on next page

given $f(0) = 1$

$f'(0) = 2$

$f''(0) = 3$

$$\lim_{x \to 0} \frac{5f(ax) + 7b\,f(x) - 12\,f(0)}{x^2}$$

to be finite; $5f(ax) + 7b\,f(x) + 12f(0) = 0$ as $x \to 0$

so $5(1) + 7b(1) - 12(1) = 0$

$7b = 12 - 5 = 7$

$b = 1$

use L'Hopital's

$$\lim_{x \to 0} \frac{5a\,f'(ax) + 7b\,f'(x)}{2x}$$

to be finite; $5a\,f'(ax) + 7b(f'(x)) = 0$ as $x = 0$

$5a(2) + 7(1)(2) = 0$

$10a = 14$

use L'Hopital's again: $a = \frac{-14}{10}$

We have values of a and b now to find limit

$$\lim_{x \to 0} \frac{5a^2\,f''(ax) + 7b\,f''(x)}{2}$$

$$= \frac{5\left(\frac{-14}{10}\right)^2(3) + 7(3)}{2}$$

$\to$ finite value

#5 $f(x) = 2x^3 - 9x^2 + 12x - 4$

$f'(x) = 6x^2 - 18x + 12$

$f''(x) = 12x - 18$

$f'(x) = 0 \Rightarrow 6x^2 - 18x + 12 = 0 = x^2 - 3x + 2 = (x-2)(x-1) \Rightarrow x = 1 \text{ or } x = 2$

$f'(x)$: $+$ | 1 | $-$ | 2 | $+$

$f''(x) = 0 \Rightarrow 12x - 18 = 0 = 2x - 3 \Rightarrow x = 3/2$

$f''(x)$: $-$ | $3/2$ | $+$

(a) increasing if $f'(x) > 0 \Rightarrow x < 1$ or $x > 2$

(b) decreasing if $f'(x) < 0 \Rightarrow 1 < x < 2$

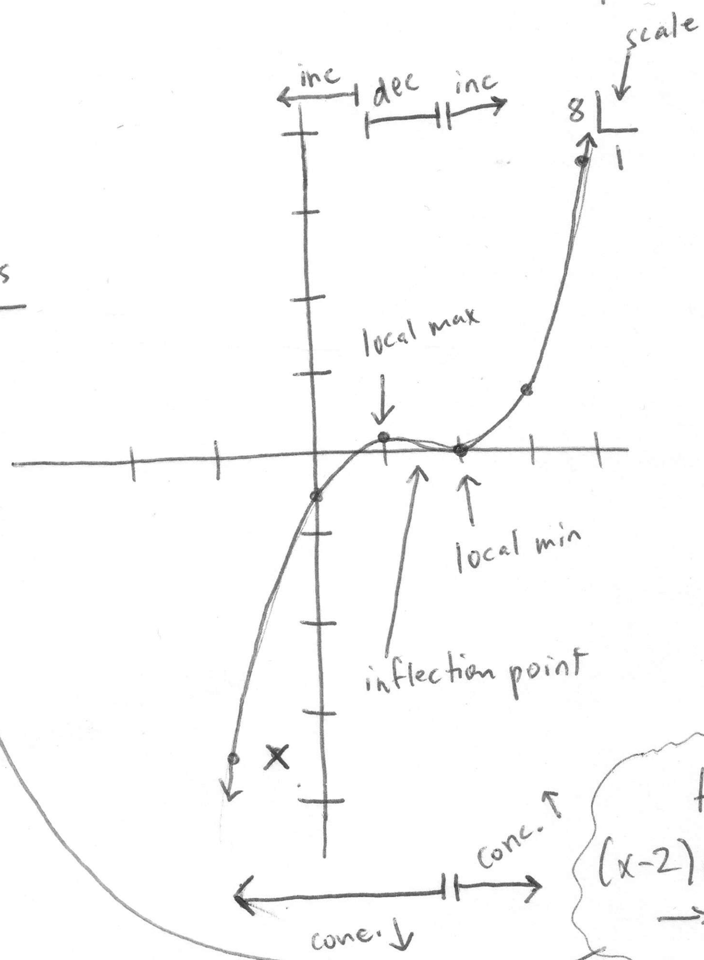
(c) concave up if $f''(x) > 0 \Rightarrow x > 3/2$

(d) concave down if $f''(x) < 0 \Rightarrow x < 3/2$

inflection pt: $x = 3/2$

local max: 1

local min: 2



2 solutions

$x = 2, x = \frac{1}{2}$

$x=2$ is a "double root"

plot some points

x	f(x)
-2	-80
-1	-27
0	-4
1	1
2	0
3	5
4	28

factor:

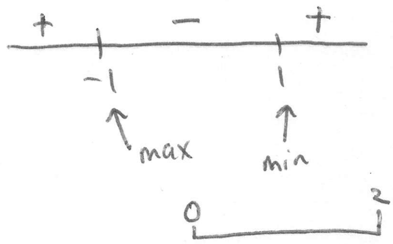
$(x-2)(2x^2 - 5x + 2)$

$\rightarrow (x-2)(x-2)(2x-1)$

#6 (a) $f(x) = x^3 - 3x + 1$ $\quad f'(x) = 3x^2 - 3$ $\quad f''(x) = 6x$

$f'(x) = 0$ if $x = \pm 1$

The smallest and largest values happen at extrema <u>or</u> on the boundary.



$f(0) = 1 \quad f(1) = -1$

$f(2) = 3 \quad \Rightarrow$ smallest value: -1

largest value : 3

(b) $f(x) = \tan^{-1}x + \frac{4}{x+2}$, $x > -2$

(i) $f'(x) = \frac{1}{1+x^2} + (4)(-1)(x+2)^{-2} = \frac{1}{1+x^2} - \frac{4}{(x+2)^2}$

$f'(x) = 0 = \frac{1}{1+x^2} - \frac{4}{(x+2)^2} \Rightarrow \frac{1}{1+x^2} = \frac{4}{(x+2)^2} \Rightarrow (x+2)^2 = 4(1+x^2) \Rightarrow x^2 + 4x + \cancel{4} = 4x^2 + \cancel{4}$

$\Rightarrow 3x^2 - 4x = 0 \Rightarrow x(3x-4) = 0 \Rightarrow \boxed{x = 0 \text{ or } x = \frac{4}{3}}$ [extrema]

(ii) vertical asymptotes: none

on $x > -2$, $f(x)$ is always defined

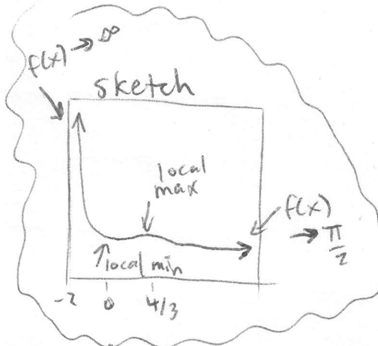
horizontal asymptotes: $\lim_{x\to\infty} \tan^{-1}(x) + \frac{4}{x+2} = \lim_{x\to\infty} \tan^{-1}(x) + \lim_{x\to\infty} \frac{4}{x+2}$

$= \frac{\pi}{2} + 0 = \frac{\pi}{2}$ $\quad$ so $\boxed{y = \frac{\pi}{2}}$

(iii) so we have $f'(x)$:

$\lim_{x\to -2^-} f(x) = \infty$, $f(0) = 2$, $f(4/3) \approx 2.13$, $\lim_{x\to\infty} f(x) = \frac{\pi}{2}$

local min $\qquad$ local max



There is no absolute max or absolute min.

As x approaches -2 from the left, $f(x)$ gets bigger and bigger.

There is no point that is a max.

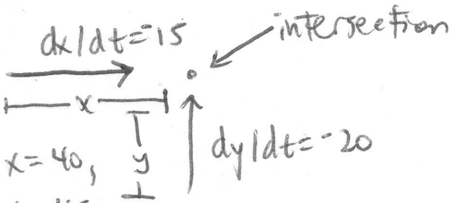
As x approaches ∞, $f(x)$ gets smaller (closer to $\pi/2$)

There is no point that is a min.

Between -2 and ∞, $f(x)$ is always between $\frac{\pi}{2}$ and ∞

#7

(a)



at $t=0$, $x=40$, $y=45$

generally, distance between them: z

$z^2 = x^2 + y^2$, $x = 40 - 15t$, $y = 45 - 20t$

we want to minimize z, so let's minimize z^2 (same thing)

$x^2 + y^2 = z^2 \Rightarrow$ minimize $x^2+y^2 \Rightarrow$ minimize $(40-15t)^2 + (45-20t)^2$

$= 25([9-4t]^2 + [8-3t]^2) = 25(81 + 16t^2 - 72t + 64 + 9t^2 - 48t)$

$= 25(25t^2 - 120t + 145) = 125(5t^2 - 24t + 29) = z(t)$

$z'(t) = 125(10t - 24)$ This equals zero when $t = 2.4$

so z reaches a minimum at $\boxed{2.4s = t}$

what is this min?? well $z^2 = (40-15t)^2 + (45-20t)^2$

$= (\underset{\uparrow\, x}{40-36})^2 + (\underset{\uparrow\, y}{45-48})^2 = 16 + 9 = 25$ $\boxed{\text{so } z = 5m}$

(b) well: $x^2 + y^2 = z^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$\Rightarrow 2(4)(-15) + 2(-3)(-20) = 2(5)\frac{dz}{dt}$

$\Rightarrow -120 + 120 = 10\frac{dz}{dt} \Rightarrow \boxed{\frac{dz}{dt} = 0}$

This makes sense! When z is at a min, $z'(t)$ should be zero!