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Research Statement

My main fields of interest are combinatorics and graph theory, but I am also interested in ordered sets, lattices, and coding theory. A primary goal of my research is to better understand the nature of extremal results typified by the following theorem of Paul Turan:

Turan's Theorem: Any graph with $2k$ vertices and more than k^2 edges must contain a set of three mutually adjacent vertices. The only graph with $2k$ vertices and k^2 edges which does not have three mutually adjacent vertices is the complete bipartite graph with two groups of k vertices. $\diamond$

A sophisticated generalization of the Pigeonhole Principle due to Frank Ramsey provides the opportunity to explore such results. The Ramsey Number $R(m,n)$ is the least positive integer such that every graph with $R(m,n)$ vertices contains either m mutually adjacent vertices or n mutually non-adjacent vertices. The largest graphs which contain neither m mutually adjacent vertices nor n mutually non-adjacent vertices for fixed values of m and n are very symmetric and beautiful, but cannot generally be constructed with algebraic objects such as finite fields. Moreover, there may be many such graphs, instead of a unique graph as in Turan's Theorem. Using a local counting argument I was able to determine when such graphs can not exist, hence establishing the best upper bounds for many small Ramsey Numbers. Please see the dynamic survey of Ramsey Numbers in the *Electronic Journal of Combinatorics* (www.combinatorics.org/Surveys/ds1.pdf).

I am now focusing my attention toward the explicit construction of extremal objects in other combinatorial problems. In 1930 Keller conjectured that any tiling of R^n with translates of the unit cube must contain a pair of cubes which share a complete $n-1$-dimensional face. Perron proved in 1940 that all cube tilings in dimensions 1 through 6 have this face sharing property. Lagarias and Shor disproved Keller's Conjecture in 1992 by constructing a cube tiling of R^{10} with no 9-dimensional face sharing. I recently found a cube tiling of R^8 with no 7-dimensional face sharing.

It remains only to determine whether such cube tilings exist in R^7. By enumerating all inequivalent local tilings which do not have face sharing in dimension 5, I seek to extend the methods of Perron to either construct a tiling of R^7 with no 6-dimensional face sharing or show that no such tilings exist. These tilings are of interest in the theory of communication because they yield effective quaternary codes.

Another such extremal problem is to determine the minimal number of crossings necessary when a graph is drawn in the plane. Richard Guy and Paul Turan conjectured the crossing numbers of K_n and $K_{m,n}$ to be $\frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$ and $\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$, respectively. They also gave constructions which show these quantities to be upper bounds for the respective crossing numbers. Researchers have verified the conjectures to be correct for small values of m and n. In particular, we know that Guy's conjecture for the complete graph is correct for $n \leq 10$. Unfortunately, the definition of equivalent drawings used by Guy and others is not strong enough to catalog and store them on the computer. I have developed a stronger definition of equivalence which should allow a computer program to generate all optimal drawings of K_n for $n \leq 11$, thus verifying the conjecture for the complete graph for $n \leq 12$.

Graph coloring problems also interest me. I formulated the following question for an undergraduate math major here at Harvard: Let G be a simple graph with no isolated vertices. When can the vertices of G be assigned positive integers so that a vertex of degree d receives an integer less than or equal to d and no two adjacent vertices receive the same integer? This problem turns out to be quite substantial. We have solved it for trees and are looking forward to answering the question for general graphs.

A final problem which indicates the direction of my research program is to find, for given m and n, the smallest number of monochromatic two by two submatrices when the entries of an m by n matrix are each independently assigned one of two colors. A computer program has shown that for small values of $m = n$ the extremal matrices have as near to half their entries of each color. Techniques used in my thesis to bound Ramsey Numbers might be adapted to prove that this must always be the case and to find the minimum number of monochromatic submatrices.