Math 19. Name______________

Mathematical Modeling
Exam II—Fall 2004
T. Judson

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Problem Number	Possible Points	Score
1	20	
2	10	
3	15	
4	15	
5	10	
6	10	
7	10	
8	10	
Total	100	

Directions—Please Read Carefully! You have two hours to take this midterm. Make sure to use correct mathematical notation. Any answer in decimal form must be accurate to three decimal places, unless otherwise specified. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You do not have to answer the problems in any particular order, so move to another problem if you find you are stuck or spending too much time. To receive full credit on a problem, you will need to justify your answers carefully—unsubstantiated answers will receive little or no credit (except if the directions for that question specifically say no justification is necessary, such as in a True/False section). Please be sure to write neatly—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers. You may use a calculator on this exam, and you are permitted one 4×6 inch note card. No other aids are allowed. ***Good Luck!!!***

1. (20 points) Consider the predator-prey system

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{6}(x-1)\left(5-x-\frac{6y}{x}\right)\\ \frac{dy}{dt} &= \frac{y}{20}\left(1-\frac{y}{x-1}\right).\end{aligned}$$

(a) Find the nullclines of the system.

(b) Find all equilibrium points of this system.

(c) Find an repelling equilibrium point for the system.

(d) Make a careful argument to show that $1 \leq x \leq 5$ and $0 \leq y \leq 6$ is a basin of attraction and use the Poincaré-Bendixson Theorem to show that the system has a periodic solution

2. (10 points) Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u, \tag{1}$$

where $0 \leq x \leq \pi$ and $t \geq 0$.

(a) Show that $u_n(t, x) = e^{-nt} \sin(\sqrt{1+n}x)$ is also a solution to equation (1) for $n = 1, 2, 3, \ldots$.

(b) Use the Principle of Superposition and the functions $u_n(x)$ to construct a solution to equation (1) that satisfies the boundary and initial conditions

$$u(0, x) = 5\sin 2x - 2\sin 3x + \sin 4x,$$
$$u(t, 0) = u(t, \pi) = 0.$$

3. (15 points) Consider the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 10u.$$

(a) Use the separation of variables technique to find nonzero solutions to this equation subject to the conditions for all $t \geq 0$ and for $0 \leq x \leq L$ such that

$$\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, L) = 0.$$

(b) Find a condition on L that guarantees a bounded solution with respect to time. That is, show that there is a solution $u = u(t, x)$ to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 10u$$

for all $t \geq 0$ and for $0 \leq x \leq L$ with

$$\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, L) = 0.$$

such that

$$\lim_{t \to \infty} u(t, x) < \infty.$$

4. (15 points) Consider the differential equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + 3\frac{\partial u}{\partial x} - 16u^3.$$

(a) Make the traveling wave substitution $u(t, x) = f(x - ct)$, where $c > 0$ is a constant to derive a differential equation in one variable, say s, for the function f.

(b) Rewrite the equation that you derived in part (a) as a pair of autonomous differential equations in the variables f and p, where both of these functions depend on the independent variable s.

(c) Sketch the phase plane for the system that you derived in part (b). Be sure to label the f and p nullclines, the equilibrium points, and mark the nullclines with arrows to indicate the direction of the trajectories the cross them.

5. (10 points) Consider the equilibrium solution $u_e = -7$ to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(u+7)$$

satisfying the boundary conditions

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,L) = 0.$$

Determine the stability of this solution.

6. (10 points) Given the equation

$$\lambda g = \frac{d^2 g}{dx^2} - \frac{1}{1+e^x} g,$$

use the Maximum Principle to determine if there is a positive number λ and a solution g defined on the interval $0 \leq x \leq L$ that vanishes at $x = 0$ and $x = L$ and is not identically zero. In particular, write *no* and justify your answer using the Maximum Principle or explain why the Maximum Principle cannot be applied.

7. (10 points)

(a) Consider an equation, $x(t)$, as a function of time, of the form

$$\frac{dx}{dt} = x^4 - 8x^2 + c,$$

where c is a constant. Find all values of c where the number of equilibrium solutions changes.

(b) Given the system

$$\begin{aligned} x' &= 0.01x - 0.003xy \\ y' &= y - 0.014xy, \end{aligned}$$

decide which equation represents the slow subsystem and which equation represents the fast subsystem. You may assume that $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$.

8. (10 points) Find a function $u(t, x)$ that solve the advection equation

$$\frac{\partial u}{\partial t} = -2\frac{\partial u}{\partial x} + 5u$$

and satisfies the initial condition

$$u(0, x) = e^{-x^2/2}.$$