key

Math 19 Problem Set #14: p. 316-317 Ex: 1-4, 9ac

(3b) $f(u) = r_1 u - r_2 u^2$ (4) $\mu \frac{d^2 u_e}{dx^2} + f(u_e) = 0$ (5) $\frac{du_e}{dx} = 0$ @ $x=0, x=L$

1. $r_1 = r_2 = 1$; $u_e = 1$

$f(u) = u - u^2$ $f'(u) = 1 - 2u$ $f'(u_e) = -1$

$\lambda g = \mu \frac{d^2 g}{dx^2} + f'(u_e) g$ $\lambda g = \mu \frac{d^2 g}{dx^2} - g$

$\frac{d^2 g}{dx^2} = \frac{\lambda + 1}{\mu} g$ $\mu > 0$ let $\frac{\lambda+1}{\mu} = c$

$c<0$: $\lambda + 1 < 0$ $\lambda < -1$ no. λ must be ≥ 0

$c=0$: $\lambda + 1 = 0$ $\lambda = -1$ no.

$c<0$: $\lambda + 1 > 0$ $\lambda > -1$ ok.

$g = \alpha e^{\sqrt{c}x} + \beta e^{-\sqrt{c}x}$

$g' = \sqrt{c}(\alpha e^{\sqrt{c}x} - \beta e^{-\sqrt{c}x})$

$g'(0) = \sqrt{c}(\alpha - \beta) = 0$ $\alpha = \beta$

$g'(L) = \sqrt{c}\alpha(e^{\sqrt{c}L} - e^{\sqrt{c}L}) = 0$ $\alpha = 0 = \beta$

no (λ, g) exist $\Rightarrow$ $\boxed{\text{stable}}$

2. $r_1 = r_2 = 1$; $u_e = 0$

$f(u) = u - u^2$ $f'(u) = 1 - 2u$ $f'(u_e) = 1$

$\lambda g = \mu \frac{d^2 g}{dx^2} + g$ $\frac{d^2 g}{dx^2} = \frac{\lambda - 1}{\mu} g$ $c = \frac{\lambda - 1}{\mu}$ $\mu > 0$

$c<0$: $\lambda - 1 < 0$ $\lambda < 1$ ok

$c=0$: $\lambda - 1 = 0$ $\lambda = 1$ ok

$c>0$: $\lambda - 1 > 0$ $\lambda > 1$ ok

if $c=0$: $g = \alpha + \beta x$ $g' = \beta$

$g'(0) = \beta = 0$ $g'(L) = \beta = 0$ $g(x) = \alpha = \mathbb{R}$

$(\lambda, g) = (1, \alpha)$ $\Rightarrow$ $\boxed{\text{unstable}}$

3. $r_1 = 1$, $r_2 = -1$; $u_e = -1$

$f(u) = u + u^2$ $f'(u) = 1 + 2u$ $f'(u_e) = -1$

$\lambda = \mu \frac{d^2 g}{dx^2} - g$ $\frac{d^2 g}{dx^2} = \frac{\lambda - 1}{\mu} g$

same as #1 $\rightarrow$ $\boxed{\text{stable}}$

4. $r_1 = 1$, $r_2 = -1$; $u_e = 0$

$f(u) = u + u^2$ $\quad f'(u) = 1 + 2u$ $\quad f'(u_e) = 1$

$\lambda g = \mu \frac{d^2 g}{dx^2} + g$ $\quad \frac{d^2 g}{dx^2} = \frac{\lambda - 1}{\mu} g$

same as #2 $\rightarrow$ $\boxed{\text{unstable}}$

9. $u(t,x)$ solves $\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2} + ru$

(a) $\frac{\partial u}{\partial t} = 0$ (time independent), $u = 0$ at $x = 0$ & $x = 10$

$0 = 2 \frac{\partial^2 u}{\partial x^2} + ru$

$\frac{\partial^2 u}{\partial x^2} = -\frac{r}{2} u$ $\quad c = -\frac{r}{2}$

$c > 0$: $u(x) = \alpha e^{\sqrt{c}x} + \beta e^{-\sqrt{c}x}$

$u(0) = \alpha + \beta = 0$ $\quad \alpha = \beta$

$u(10) = \alpha (e^{\sqrt{c}10} - e^{-\sqrt{c}10}) = 0$ $\quad \alpha = 0 = \beta$

$c = 0$: $u(x) = \alpha + \beta x$

$u(0) = \alpha = 0$

$u(10) = \beta(10) = 0$ $\quad \beta = 0$

$c < 0$: $u(x) = \alpha \cos(\sqrt{-c}x) + \beta \sin(\sqrt{-c}x)$

$u(0) = \alpha = 0$

$u(10) = \beta \sin(\sqrt{-c}10) = 0$

$\beta = 0$ or $\beta = \mathbb{R}$ & $\sqrt{-c}10 = n\pi$ $\quad \sqrt{\frac{r}{2}} = \frac{n\pi}{10}$ $\quad \boxed{r = \frac{n^2\pi^2}{50}}$

(b) same as (a)

(c) $\frac{\partial u}{\partial t} = 0$, $u = 0$ at $x = 0$ & $x = 20$

$0 = 2 \frac{\partial^2 u}{\partial x^2} + ru$

$\frac{\partial^2 u}{\partial x^2} = -\frac{r}{2} u$ $\quad c = -\frac{r}{2}$

$\left.\begin{array}{l} c > 0: \alpha = 0 = \beta \\ c = 0: \alpha = 0 = \beta \end{array}\right\}$ see (a)

$c < 0$: $u(x) = \alpha \cos(\sqrt{-c}x) + \beta \sin(\sqrt{-c}x)$

$u(0) = \alpha = 0$

$u(20) = \beta \sin(\sqrt{-c}20) = 0$

$\beta = 0$ or $\beta = \mathbb{R}$ & $\sqrt{-c}20 = n\pi$ $\quad \boxed{r = \frac{n^2\pi^2}{200}}$