Exam Review Notes

November 2, 2004

General Info about the exam and the type of questions that could/will not be asked

- The midterm will be held in Science Center D, Wed. 7:30-9:30
- Calculators permitted at the exam
- There will be no exercises involving eigenvectors/eigenvalues
- There is a high probability for the exam to contain problems involving
  - the computation of Partial Derivatives
  - matrix Operations
  - matching some given differential equations with their slope fields (graphs will be given)

  Practice them!
- The principle of superposition states the following: If you have the system

$$\begin{cases} x' = 2x - 5y \\ y' = 3x + 2y \end{cases}$$

  and its solutions are $\bar{x_1} = \binom{x_1(t)}{y_1(t)}$ and $\bar{x_2} = \binom{x_2(t)}{y_2(t)}$ then any linear combination of these two solutions of the form $a \cdot \bar{x_1} + b \cdot \bar{x_2}$, where a and b are constants, will also be a solution for this system.

Practice Problems solved during the Review Session

Problem 1

We are given the following system:

$$\begin{cases} x' = x - 3x^2 + xy \\ y' = x - xy \end{cases}$$

This is a non-linear system. Analyze it using all the various tools we have learned in class:

a) Find the x and y nullcines:

Finding the x-nullclines: $0 = x' = x - 3x^2 + xy \quad \Rightarrow \quad x(1 - 3x + y) = 0$

Thus the 2 x-nullclines are: $x = 0$ and $y = 3x - 1$

Finding the y-nullclines: $0 = y' = y - xy \quad \Rightarrow \quad y(1 - x) = 0$

Thus the 2 y-nullclines are: $x = 1$ and $y = 0$

b) Graph the x and y nullclines, perform the phase-plane analysis for this system (find the signs of dx/dt and dy/dt on the different sides of the nullclines, mark the direction of movement across the nullclines, mark the general direction of movement in the regions separated by the nullclines), and find the equilibrium points.

(See graph on the next page.)

The equilibrium solutions are: (0,0), (1,2), (1/3,0).

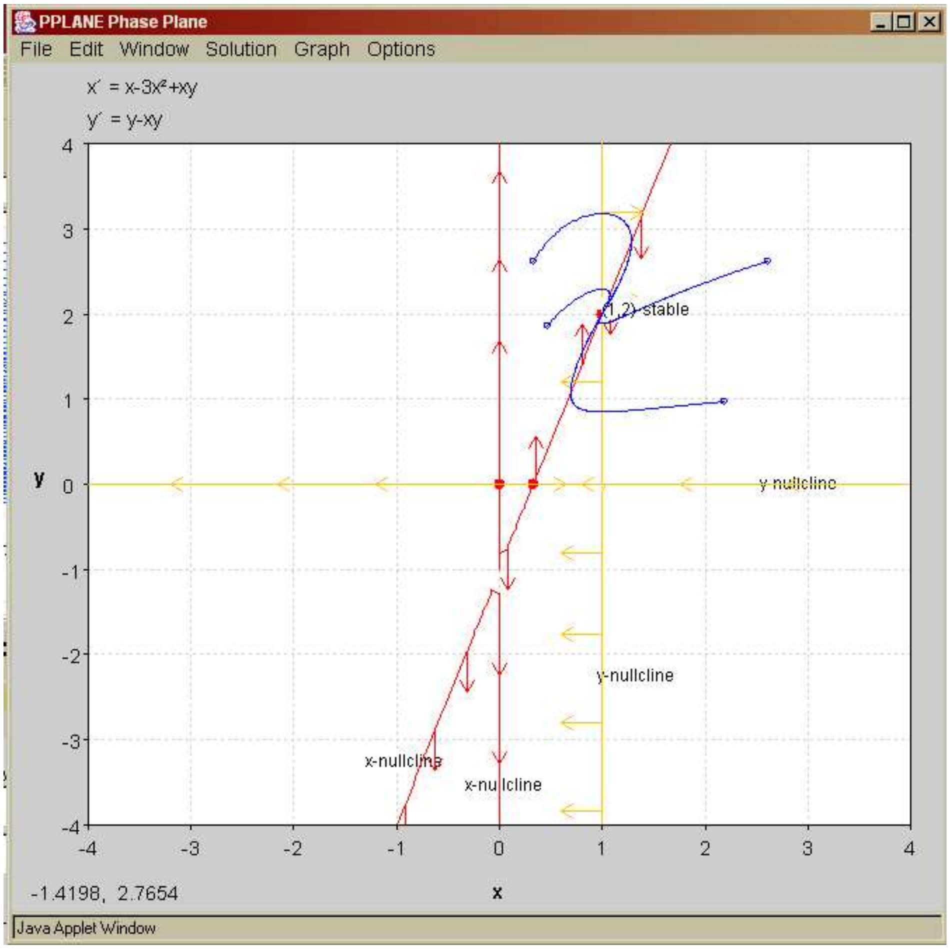
c) Analyze the stability of the equilibrium solutions.

Compute the matrix D.

$$D = \begin{pmatrix} 1 - 6x + y & x \\ -y & 1 - x \end{pmatrix}$$

Equilibrium Point	Matrix D	det(D) and tr(D)	Stability
$(0,0)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$det(D) = 1 > 0, tr(D) = 2 > 0$	Unstable
$(1,2)$	$\begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix}$	$det(D) = 2 > 0, tr(D) = -3 < 0$	Stable
$(1/3,0)$	$\begin{pmatrix} -1 & 1/3 \\ 0 & 2/3 \end{pmatrix}$	$det(D) = -2/3 < 0, tr(D) = -1/3 < 0$	Unstable

- Notice that if you have a stable equilibrium solution, a trajectory can spiral in toward this solution, or just go toward the solution point, getting arbitrarily close to it, but never actually touching it.



d) Suppose that the system we have is a preditor-prey system. Who is the preditor and who is the prey?

Since the equation for x' has xy as a positive term, we can conclude that x is the "preditor" since x benefits from the interaction between x and y (y is the "prey").

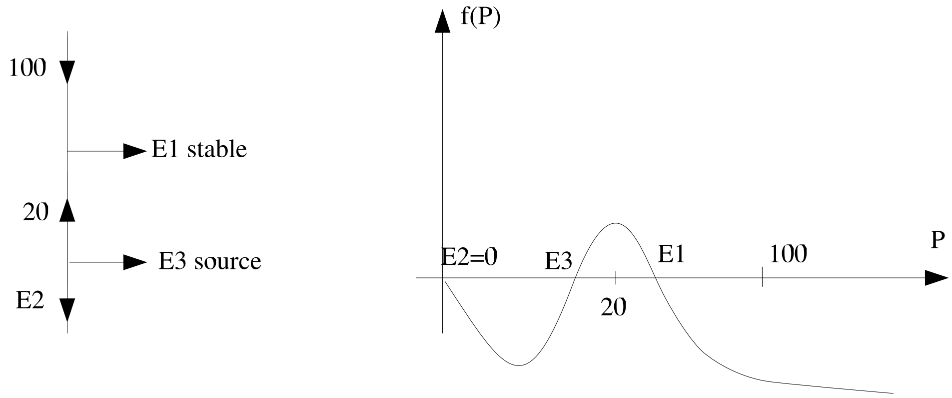
Does x has any other food source than its prey?

YES. Notice that if we ignore the xy term, we get a logistic growth equation for x.

Problem 2: Drawing a phase-line diagram

We are given the following information:

- $p = 0$ – population is constant;
- p is close to 0 – population decreases;
- $p = 20$ – population increases;
- $p > 100$ – population decreases;



Since our equilibrium points are $E_2 = 0$, E_3, and E_1, we can now write a potential differential function for this situation:
$\frac{dP}{dt} = f(P) = -P(P - E_3)(P - E_1)$

Problem 3: Preditor-Prey system

We are given the following two systems:

1. $\begin{cases} \frac{dx}{dt} = 10(1 - x/10) - 20xy \\ \frac{dy}{dt} = -5y + xy/20 \end{cases}$

2. $\begin{cases} \frac{dx}{dt} = 0.3x - xy/100 \\ \frac{dy}{dt} = 15y(1 - y/15) + 25xy \end{cases}$

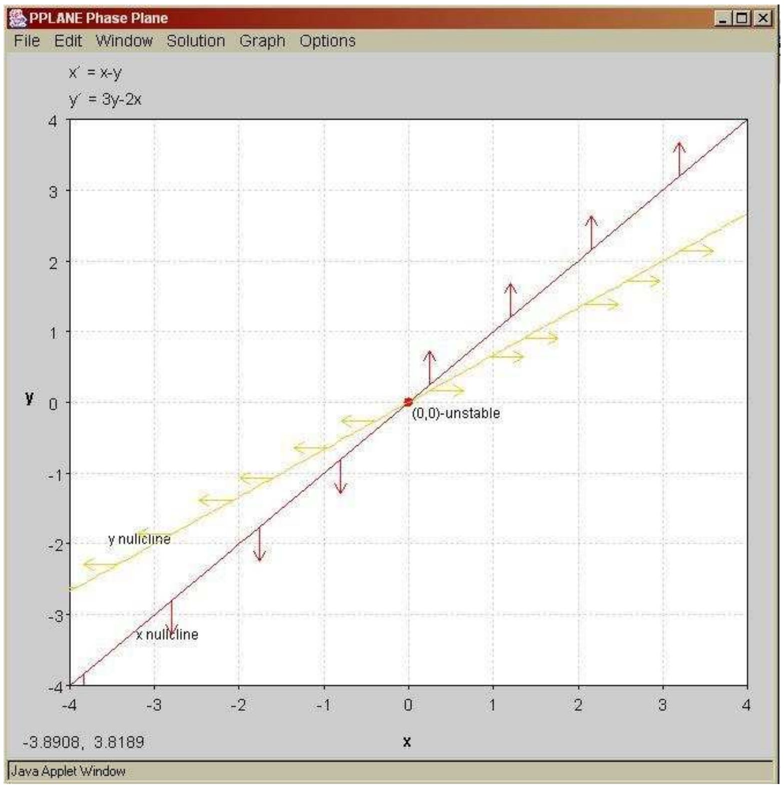
y is the preditor in both cases since the xy term in the $\frac{dy}{dy}$ equation is positive for both systems.

As an example for the first system, we can mention the "blue whale and plancton" example; and, we can associate the second system with the "elefant and mosqitoes" example since in the first system the preditor (the blue whale) benefits little from the interaction ($+xy/20$), whereas the benefit of the preditor in the second system (mosquitoes) is much greater ($+25xy$).

Problem 4: Linear systems

$$\begin{cases} \frac{dx}{dt} = x - y \\ \frac{dy}{dt} = 3y - 2x \end{cases}$$

Perform phase-plane analysis.
x-nullclines: $y = x$.
y-nullclines: $y = \frac{2}{3}x$



Find equilibrium points:
$D = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$
$det(D) = 1 > 0$.
$tr(D) = 4 > 0$.
Thus, the equilibrium point $(0, 0)$ is unstable.