Math 19. Lecture 22
Pattern Formation (II)

T. Judson

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1 Stability

Suppose that $u_e(x)$ is a solution to

$$\mu \frac{d^2 u_e}{dx^2} + f(u_e) = 0$$

subject to the boundary conditions. Let $w(x)$ be a small perturbation of $u_e(x)$ at $t = 0$, and set

$$u(0, x) = u_e(x) + w(x)$$

and move forward in time to obtain a solution to

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + f(u) \tag{1}$$

$$\frac{\partial}{\partial x} u(t, 0) = \frac{\partial}{\partial x} u(t, L) = 0. \tag{2}$$

that is equal to $u_e(x) + w(x)$ at $t = 0$

If $w(x)$ is small enough, then the resulting solution $u(t, x)$ to (1) and (2) that has the property $u(0, x) = u_e(x) + w(x)$ has the property that at *every* x, the values of $u(t, x) \to u_e(x)$ as $t \to \infty$.

A solution is unstable if there is an arbitrarily small (but not identically zero) perturbation $w(x)$ such that $u(t, x)$ does not approach $u_e(x)$ for *at least one* x as $t \to \infty$.

2 Linear Stability

This definition satisfies our intuition, but stability may be impossible to verify for a given f. We give a stronger condition for stability below, *linear stability*.

- Linear stability $\Rightarrow$ Stability
- Stability $\not\Leftarrow$ Linear stability
- Linear stability guarantees stability against slight changes in the *equation* not just slight changes in the starting function $u(0, x)$.

The definition of linear stability is somewhat technical, but it is more relevant in the real world.

We first construct a new function $z(x)$ from the function f and from the equilibrium solution $u_e(x)$ to

$$\mu \frac{d^2 u_e}{dx^2} + f(u_e) = 0$$
$$\frac{d}{dx} u_e(0) = \frac{d}{dx} u_e(L) = 0.$$

Define $z(x)$ by

$$z(x) = \left. \frac{df}{du} \right|_{u=u_e}.$$

For example, If $f(u) = r_1 u - r_2 u^2$, where $r_1, r_2 > 0$, then

$$z(x) = r_1 - 2 r_2 u_e(x).$$

3 Linear Stability Criterion

The solution $u_e(x)$ is a *stable* solution to

$$\mu \frac{d^2 u_e}{dx^2} + f(u_e) = 0$$
$$\frac{d}{dx} u_e(0) = \frac{d}{dx} u_e(L) = 0$$

if and only if there is *no* pair (g, λ) , where $g(x)$ is some function that is *not* identically zero for $0 \leq x \leq L$, where $\lambda \in \mathbb{R}$, and where the following constraints are satisfied.

- $\lambda \geq 0$
- $\lambda g = \mu \dfrac{d^2 g}{dx^2} + z(x) g$
- $\left.\dfrac{dg}{dx}\right|_{x=0} = \left.\dfrac{dg}{dx}\right|_{x=L} = 0$

A solution is *unstable* if there is even one such pair (g, λ) that obeys the above conditions.

4 An Example

Let

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 5u(2-u)$$

for all t and for $0 \leq x \leq 1$. Assume that we have boundary conditions

$$\frac{\partial}{\partial x} u(t,0) = \frac{\partial}{\partial x} u(t,1) = 0.$$

The solutions that are independent of t and x are $u = 0$ and $u = 2$.

To check stability, we ask whether there is a pair (g, λ), where $\lambda \geq 0$ and $g(x) \not\equiv 0$ and g is a solution to

$$\lambda g = \frac{d^2 g}{dx^2} + f'(u_e) g.$$

Since $f(u) = 5u(2-u)$ and $u_e = 0$ or $u_e = 2$,

$$\begin{aligned} f'(0) &= 10, \\ f'(2) &= -10. \end{aligned}$$

Also, g must satisfy

$$\left.\frac{dg}{dx}\right|_{x=0} = \left.\frac{dg}{dx}\right|_{x=1} = 0.$$

If such a (g, λ) exists, then the equilibrium solution u_e is unstable.

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology.* Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 18.
- "Dynanamics of Stripe Formation," pp. 280–282.
- "A Reaction-Diffusion Wave on the Skin of the Marine Angelfish," pp. 282–286.
- "Letters to Nature," pp. 286–288.