

# Math 19. Lecture 12

## Matrix Notation

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### 1 Matrix Basics

Recall that we can rewrite the system

$$\begin{aligned}\frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy\end{aligned}$$

using matrix notation as

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

where

$$\mathbf{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \frac{d\mathbf{x}}{dt} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}, \text{ and } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

### 2 Matrix Operations

- Addition of matrices.
- Multiplying a matrix by a scalar.
- The product of a matrix and a column vector

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = x \begin{pmatrix} a \\ c \end{pmatrix} + y \begin{pmatrix} b \\ d \end{pmatrix}$$

- The product of two matrices

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} ex + fy \\ gx + hy \end{pmatrix} \\ &= \begin{pmatrix} (ae + bg)x + (af + bh)y \\ (ce + dg)x + (cf + dh)y \end{pmatrix} \\ &= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

- $AB = (Ab_1, Ab_2)$ .
- In general,  $AB \neq BA$ .

### 3 Solving a $2 \times 2$ Linear System

Consider the system

$$\begin{aligned} \frac{dx}{dt} &= x + 2y \\ \frac{dy}{dt} &= 4x + 3y \end{aligned}$$

with initial conditions  $x(0) = 1$  and  $y(0) = 1$ . For

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

we can guess that the solution is of the form

$$\mathbf{x}(t) = \begin{pmatrix} c_1 e^{\lambda t} \\ c_2 e^{\lambda t} \end{pmatrix},$$

where  $\lambda, c_1, c_2 \in \mathbb{R}$ .

### 4 Checking the Guess

On the one hand, we have

$$\frac{d\mathbf{x}}{dt} = \lambda \begin{pmatrix} c_1 e^{\lambda t} \\ c_2 e^{\lambda t} \end{pmatrix} = \lambda \mathbf{x}.$$

On the other hand,

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} = e^{\lambda t} A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

To ensure a solution, we must choose  $\lambda, c_1, c_2$  such that

$$A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

.

## 5 Finding Eigenvalues and Eigenvectors

Let

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The system  $A\mathbf{x} = \lambda\mathbf{x}$  can be written as either

$$\begin{aligned} x_1 + 2x_2 &= \lambda x_1 \\ 4x_1 + 3x_2 &= \lambda x_2. \end{aligned}$$

We can reduce this system to

$$\begin{aligned} (1 - \lambda)x_1 + 2x_2 &= 0 \\ (\lambda^2 - 4\lambda - 5)x_2 &= 0. \end{aligned}$$

Therefore, to obtain a nonzero solution either  $\lambda = 5$  or  $\lambda = -1$ .

- If  $\lambda = 5$ , the first equation in the system becomes  $-2x_1 + x_2 = 0$ , and we can let

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

- If  $\lambda = -1$ , then

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For

$$A\mathbf{x} = \lambda\mathbf{x},$$

The number  $\lambda$  is called an *eigenvalue* of  $A$ , and  $\mathbf{x} \neq \mathbf{0}$  is an *eigenvector* corresponding to  $\lambda$ .

## 6 The Principle of Superposition

The solutions to the system

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

must be of the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ or } \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \beta e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The *Principle of Superposition* tells us that any linear combination of solutions to a linear equation is also a solution. Therefore

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \alpha e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus,

$$\begin{aligned} x(t) &= \frac{2}{3}e^{5t} + \frac{1}{3}e^{-t} \\ y(t) &= \frac{4}{3}e^{5t} - \frac{1}{3}e^{-t}. \end{aligned}$$

### Homework

- Chapter 11. Exercises 1, 2, 3, 4; p. 177.

### Readings and References

- C. Taubes. *Modeling Differential Equations in Biology*. Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 11.