

Math 19. Lecture 7

Phase Plane Analysis

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1 An Epidemic Model

Consider the model of a viral epidemic that moves through an isolated population. We make the following assumptions.

- (a) Individuals are infected at a rate proportional to the product of the number of infected and susceptible individuals. We assume that the constant of proportionality is $\lambda = 0.05$ per day.
- (b) The length of the incubation period is negligible. Infectious individuals are immediately infectious.
- (c) On the average, an infected individual dies or recovers after 10 days.
- (d) No one is sick initially.
- (e) Infected individuals do not give birth, but susceptible individuals have a birth rate of 0.0003 per individual per year. Newborns are susceptible.

If $x(t)$ is the number of susceptible and $y(t)$ is the number of infected people, then

$$\begin{aligned}\frac{dx}{dt} &= -\lambda xy + 0.0003x \\ \frac{dy}{dt} &= \lambda xy - 0.1y.\end{aligned}$$

The first equation asserts that in a unit time interval, any infected individual will infect any given susceptible individual with $\lambda = 0.05$ percent probability.

The constant λ is a measure of the relative infectivity of the disease. If λ is high relative to the birth rate then the disease will burn itself out.

2 Phase Plane Analysis

- *Step 1.* Draw the curves where $f(x, y) = 0$. These curves are called the *x null clines*. When $\mathbf{v}(t)$ lies on one of these curves, $dx/dt = 0$. Draw vertical slash marks on the *x* null clines to remind yourself that a trajectory crossing the null cline can only do so if it is moving in a vertical direction at the instant of crossing.
- *Step 2.* Draw the curves where $y(x, y) = 0$. These curves are called the *y null clines*. When $\mathbf{v}(t)$ lies on one of these curves, $dy/dt = 0$. Draw horizontal slash marks on the *y* null clines to remind yourself that a trajectory crossing the null cline can only do so if it is moving in a horizontal direction at the instant of crossing.
- *Step 3.* Label the points where the *x* and *y* null clines intersect. These intersections are called *equilibrium points*. If $\mathbf{v}(t)$ is ever at one of these points, then both dx/dt and dy/dt vanish. This means that the trajectory stays at the point for all time. If the system is going to settle into a steady state, then $\mathbf{v}(t)$ will approach one of the equilibrium points as $t \rightarrow \infty$.
- *Step 4.* Label the regions of the *xy*-plane where $dx/dt < 0$ and where $dx/dt > 0$. These regions are always separated by *x* null clines. Likewise, label the regions where dy/dt is positive and negative.
- *Step 5.* Go back and put arrows on the vertical hash marks of the *x* null clines. These arrows indicate whether the motion across the null cline is up or down. The arrows are up on the parts of the *x* null cline that are in the $dy/dt > 0$ region, and down on those parts of the *x* null cline in the $dy/dt < 0$ regions. Likewise, draw arrows on the horizontal slash marks of the *y* null clines. These arrows are pointing right on the parts of the *y* null cline in the $dx/dt > 0$ regions and left point on the parts in the $dx/dt < 0$ regions.
- *Step 6.*

- (a) If $dx/dt > 0$ and $dy/dt > 0$, then both $x(t)$ and $y(t)$ are increasing and the trajectory moves up and right.
- (b) If $dx/dt > 0$ and $dy/dt < 0$, the trajectory moves down and right.
- (c) If $dx/dt < 0$ and $dy/dt > 0$, the trajectory moves up and left.
- (d) If $dx/dt < 0$ and $dy/dt < 0$, the trajectory moves down and left.

3 Existence and Uniqueness

The system of equations

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned} \tag{1}$$

is completely predictive. If you choose a starting point in the xy -plane, then there is exactly one solution to the system that starts at your chosen point.

- For any starting point in the xy -plane, there is a unique solution to (1).
- Think of $\mathbf{v}(t)$ as tracing out a trajectory in the xy -plane as t increases. The goal is to predict the behavior of this trajectory.
- The phase plane analysis is done to help predict the trajectory.

Homework

- Chapter 6. Exercises 1, 2, 3, 4 (a, c) 5; pp. 102–103.

Readings and References

- C. Taubes. *Modeling Differential Equations in Biology*. Prentice Hall, Upper Saddle River, NJ, 2001. Chapter 6.
- “Wolves, Moose, and Tree Rings on Isle Royale,” pp. 96–102.