Math 19. Mathematical Modeling
Midterm I—Fall 2005
Solutions T. Judson

1. (10 points) Consider two populations of interacting species $x = x(t)$ and $y = y(t)$. Assume that both species grows logistically in the absence of the other.

   (a) Write a system of differential equations that describes the situation when species $x(t)$ is a parasite and species $y(t)$ is a host and where the presence of the parasite is beneficial to the host.

   (b) Write a system of differential equations that describes the situation when species $x(t)$ is a parasite and species $y(t)$ is a host and where the presence of the parasite is harmful to the host.

   Solution.

   (a)

   $$\begin{aligned} x' &= (a_1 + b_1 x)x + c_1 xy \\ y' &= (a_2 + b_2 y)y + c_2 xy, \end{aligned}$$

   where a_i, b_i, and c_i are positive constants.

   (b)

   $$\begin{aligned} x' &= (a_1 + b_1 x)x + c_1 xy \\ y' &= (a_2 + b_2 y)y - c_2 xy, \end{aligned}$$

   where a_i, b_i, and c_i are positive constants.

2. (12 points) Consider the system of differential equations

   $$\begin{aligned} \frac{dx}{dt} &= -x - 2y, \\ \frac{dy}{dt} &= 4x + 3y. \end{aligned}$$

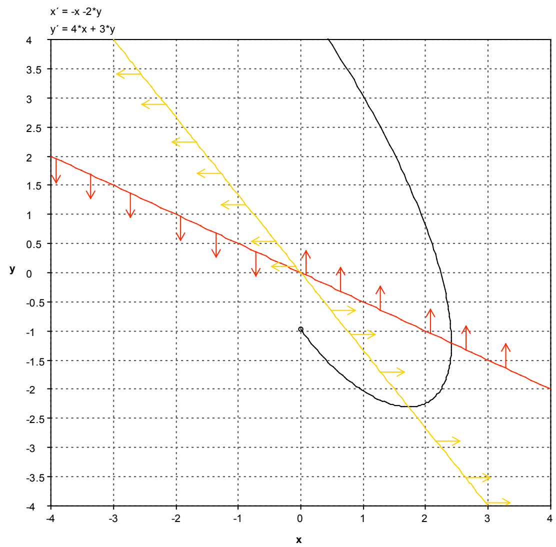
   (a) Draw and label the x and y-nullclines on the graph below. Be sure to indicate the direction of the solution on the nullclines.

   (b) Label the regions where $dx/dt > 0$ and where $dx/dt < 0$ on the graph. Do the same for dy/dt.

   (c) Decide if $(0, 0)$ is a stable equilibrium point. Justify your answer.

   (d) For the initial condition $x(0) = 0$ and $y(0) = -1$, sketch the trajectory in the phase plane on the graph on the previous page.

Solution.



The equilibrium solution is unstable since the trace of

$$\begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix}$$

is positive.

3. (20 points) Consider the system I

$$\begin{aligned} \frac{dx}{dt} &= 6x - 2x^2 - 3xy \\ \frac{dy}{dt} &= y - xy - y^2, \end{aligned}$$

(a) Find the x and y-nullclines of the system.

(b) Find all of the equilibrium solutions.

(c) Using linearization, determine the nature of each equilibrium solution.

(d) Sketch and *label* the nullclines on the graph below. Be sure to indicate the direction of the solution on the nullclines. Sketch the trajectory in xy-plane that begins at $(2, 2)$.

Solution.

Since the equilibrium solutions are at $(0, 0)$, $(0, 1)$, $(3, 0)$, and $(-3, 4)$, and

$$D(x, y) = \begin{pmatrix} \frac{\partial}{\partial x}(6x - 2x^2 - 3xy) & \frac{\partial}{\partial y}(6x - 2x^2 - 3xy) \\ \frac{\partial}{\partial x}(y - xy - y^2) & \frac{\partial}{\partial y}(y - xy - y^2) \end{pmatrix} = \begin{pmatrix} 6 - 4x - 3y & -3x \\ -y & -x - 2y \end{pmatrix}.$$

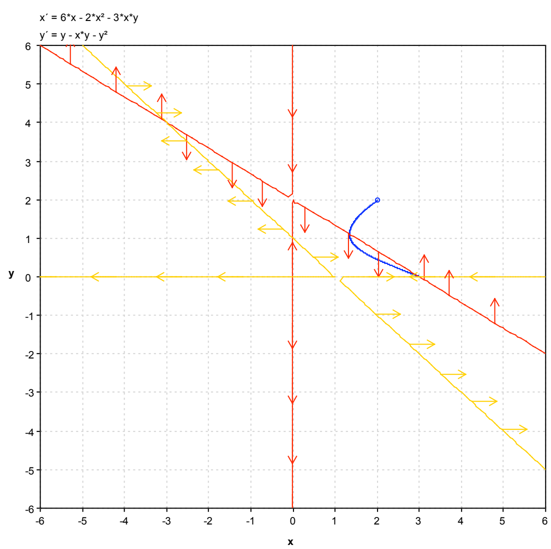
we have

$$\begin{aligned} D(0,0) &= \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \\ D(0,1) &= \begin{pmatrix} 3 & 0 \\ 0 & -6 \end{pmatrix} \\ D(3,0) &= \begin{pmatrix} -6 & -9 \\ 0 & -3 \end{pmatrix} \\ D(-3,4) &= \begin{pmatrix} 6 & 9 \\ -4 & -5 \end{pmatrix} \end{aligned}$$

The only matrix with negative trace and positive determinant is

$$D(3,0) = \begin{pmatrix} -6 & -9 \\ 0 & -3 \end{pmatrix}.$$

Hence, the only stable equilibrium solution is $(3,0)$.



4. (8 points) Match equations and slope fields

(a) $\dfrac{dy}{dt} = 1 + y^2$

(b) $\dfrac{dy}{dt} = y^2 - t^2$

(c) $\dfrac{dy}{dt} = ty$

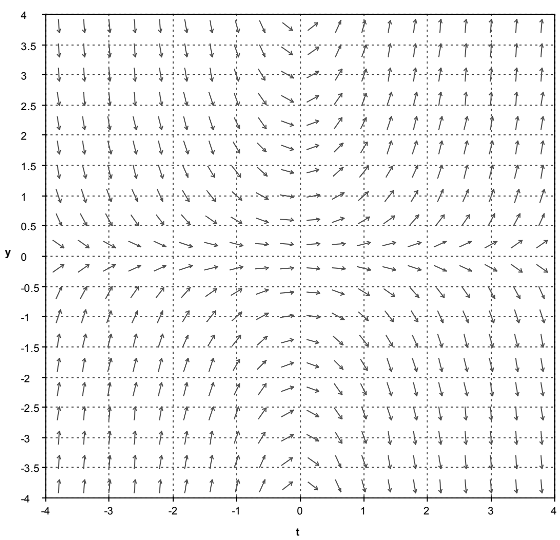
(d) $\dfrac{dy}{dt} = 1 - y$

(e) $\dfrac{dy}{dt} = y(1 - y) - 2$

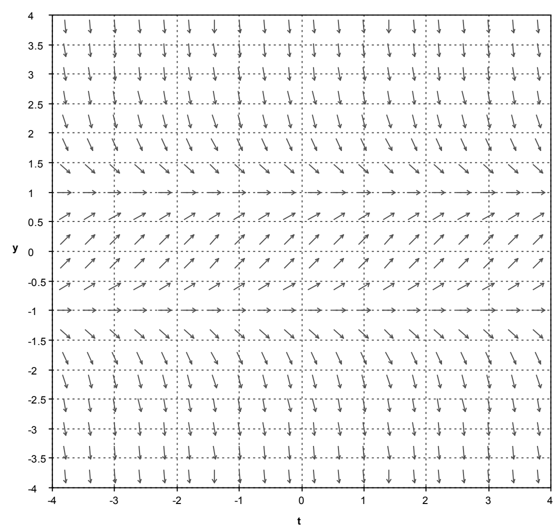
(f) $\dfrac{dy}{dt} = (y - t)^2$

(g) $\dfrac{dy}{dt} = 1 - y^2$

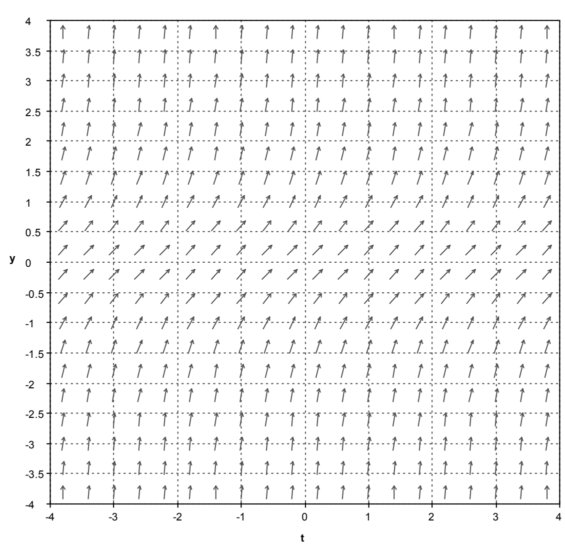
(h) $\dfrac{dy}{dt} = y^2 - t^2$



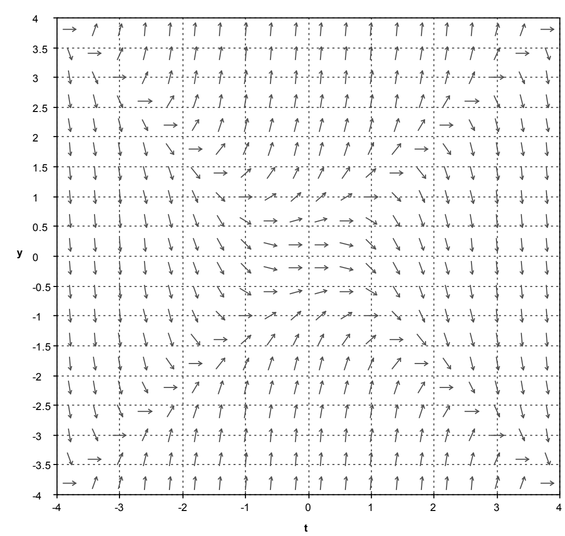
(i)



(ii)



(iii)



(iv)

Solution. (i) c, (ii) g, (iii) a, (iv) b or h.

5. (12 points) Compute each of the following partial derivatives:

(a) $\dfrac{\partial}{\partial x}(x^2 + \sin y + xe^y)$

(b) $\dfrac{\partial^2}{\partial x \partial y}(x^2 + \sin y + xe^y)$

(c) $\dfrac{\partial^2}{\partial x^2}(x^2 + \sin y + xe^y)$

(d) $\dfrac{\partial^2}{\partial y^2}(x^2 + \sin y + xe^y)$

Solution.

(a) $\dfrac{\partial}{\partial x}(x^2 + \sin y + xe^y) = 2x + e^y$

(b) $\dfrac{\partial^2}{\partial x \partial y}(x^2 + \sin y + xe^y) = e^y$

(c) $\dfrac{\partial^2}{\partial x^2}(x^2 + \sin y + xe^y) = 2$

(d) $\dfrac{\partial^2}{\partial y^2}(x^2 + \sin y + xe^y) = -\sin y + xe^y$

6. (8 points) Compute each of the following definite integrals:

(a) $\displaystyle\int_0^2 \int_0^1 4x^3 + 6xy^2 \, dxdy$

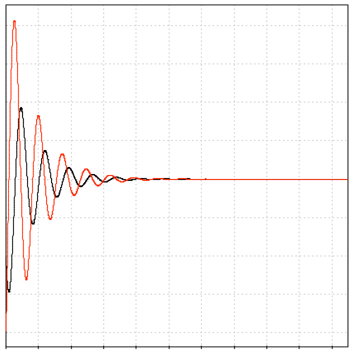
(b) $\displaystyle\int_0^1 \int_0^{\pi/2} e^y + \sin x \, dxdy$

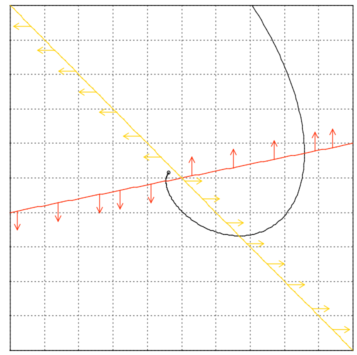
Solution.

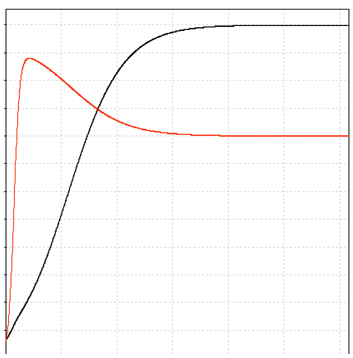
(a) $\int_0^2 \int_0^1 4x^3 + 6xy^2 \, dxdy = \int_0^2 \left[x^4 + 3x^2y^2\right]_0^1 \, dy = \int_0^2 1 + 3y^2 \, dy = 10$

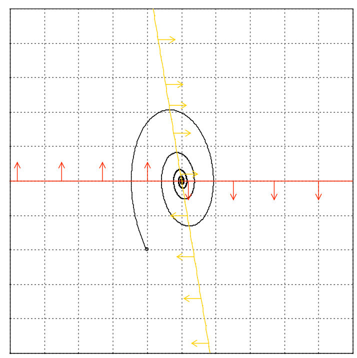
(b) $\int_0^1 \int_0^{\pi/2} e^y + \sin x \, dxdy = \int_0^1 [xe^y - \cos x]_0^{\pi/2} \, dy = \int_0^1 1 + \pi e^y/2 \, dy = [y + \pi e^y/2]_0^1 = 1 + \pi(e-1)/2$

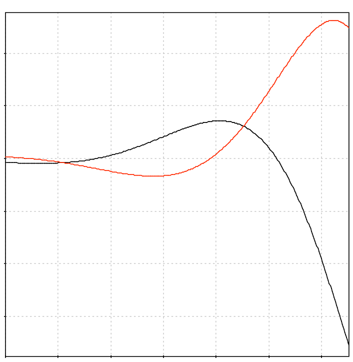
7. (8 points) In the following images, match the $t \to (t, x(t))$ and $t \to (t, y(t))$ plots in the first column with the corresponding phase plane plots, $t \to (x(t), y(t))$ in the second column.

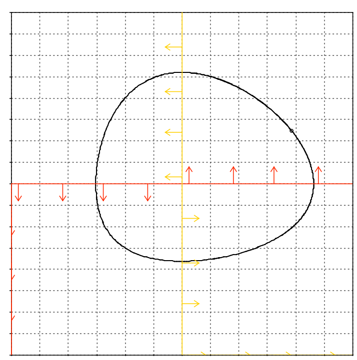


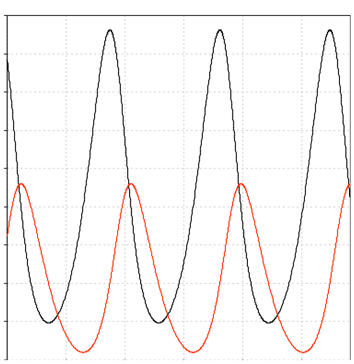


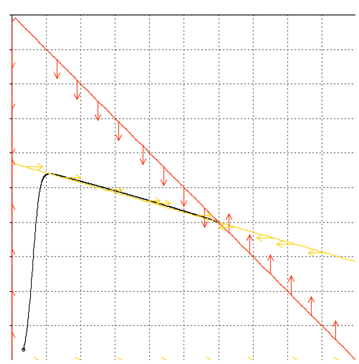




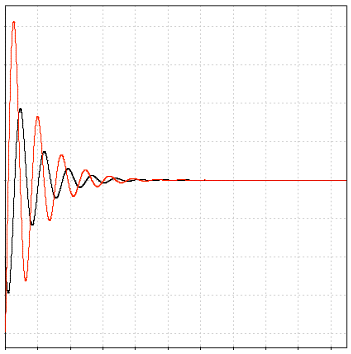
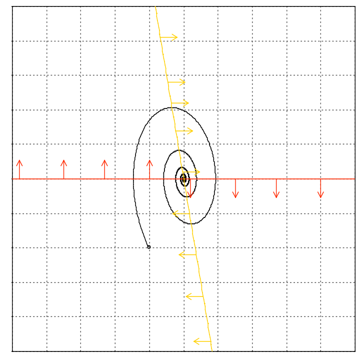


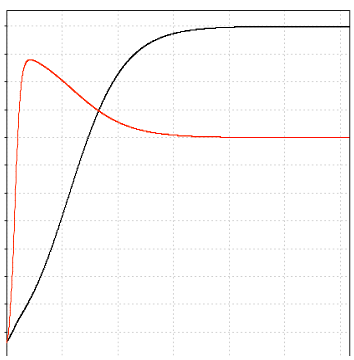
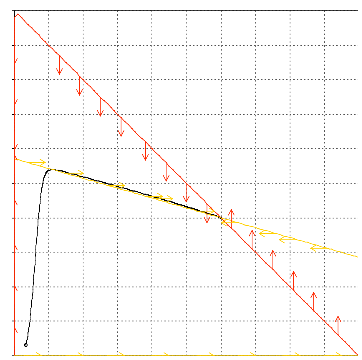


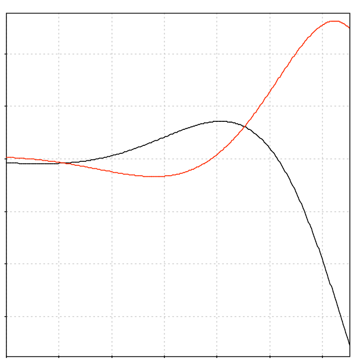
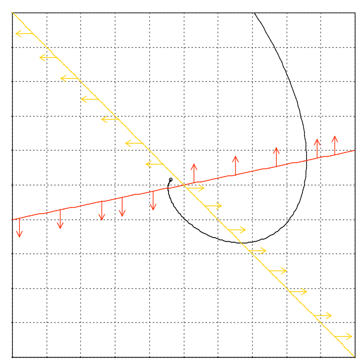


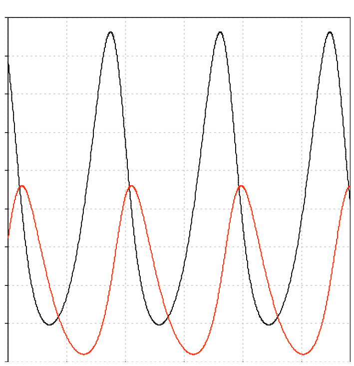
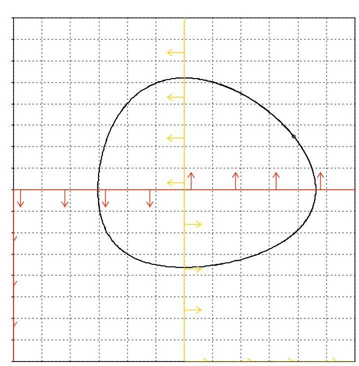


Solution.

 ⟶ 

 ⟶ 

 ⟶ 

 ⟶ 

8. (12 points)

(a) Sketch a graph of $f(y) = (y+1)(y-9)^2$

(b) Use the graph in part (a) to develop a phase line for the autonomous differential equation

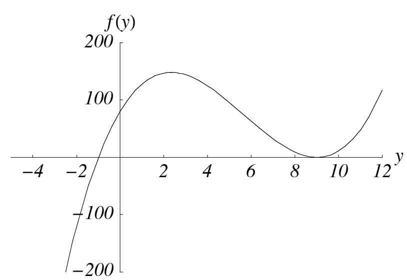
$$y' = (y+1)(y-9)^2.$$

(c) Classify each equilibrium solution of the differential equation $y' = (y+1)(y-9)^2$ as a source, a sink, or a node.

(d) Sketch the equilibrium solutions of $y' = (y+1)(y-9)^2$ in the ty-plane. These equilibrium solutions divide the ty-plane into regions. Sketch at least one solution trajectory in each of the regions.

Solution.

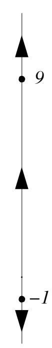
(a) Sketch a graph of $f(y) = (y+1)(y-9)^2$



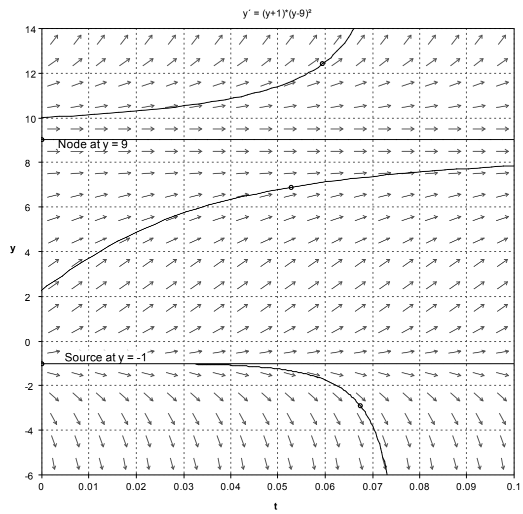
(b) Use the graph in part (a) to develop a phase line for the autonomous differential equation

$$y' = (y+1)(y-9)^2.$$

(c) Classify each equilibrium solution of the differential equation $y' = (y+1)(y-9)^2$ as a source, a sink, or a node.



(d) Sketch the equilibrium solutions of $y' = (y+1)(y-9)^2$ in the ty-plane. These equilibrium solutions divide the ty-plane into regions. Sketch at least one solution trajectory in each of the regions.



9. (10 points) Suppose that you wish to model a population with a differential equation of the form $dP/dt = f(P)$, where $P(t)$ is the population at time t. Experiments have been performed on the population that give the following information:

- The population at $P = 0$ remains constant.
- A population close to 0 will decrease.
- A population of $P = 20$ does not change.
- A population of $20 < P < 100$ increases.
- A population of $P > 100$ will decrease.

(a) Sketch the simplest possible phase line that agrees with the experimental information above.

(b) Write a formula for $f(P)$ that agrees with the phase line in part (a).

Solution. $\dfrac{dP}{dt} = f(P) = kP(100 - P)(P - 20)$