Math 19. **Name________________**

Mathematical Modeling
Midterm I—Fall 2005
T. Judson

Do not write in this space.

Problem Number	Possible Points	Score
1	10	
2	12	
3	20	
4	8	
5	12	
6	8	
7	8	
8	12	
9	10	
Total	84	

Directions—Please Read Carefully! You have two hours to take this midterm. Make sure to use correct mathematical notation. Any answer in decimal form must be accurate to three decimal places, unless otherwise specified. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You do not have to answer the problems in any particular order, so move to another problem if you find you are stuck or spending too much time on a single problem. To receive full credit on a problem, you will need to justify your answers carefully—unsubstantiated answers will receive little or no credit (except if the directions for that question specifically say no justification is necessary, such as a True/False section). Please be sure to write neatly—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers. No calculators allowed. **Good Luck!!!**

1. (10 points) Consider two populations of interacting species $x = x(t)$ and $y = y(t)$. Assume that both species grows logistically in the absence of the other.

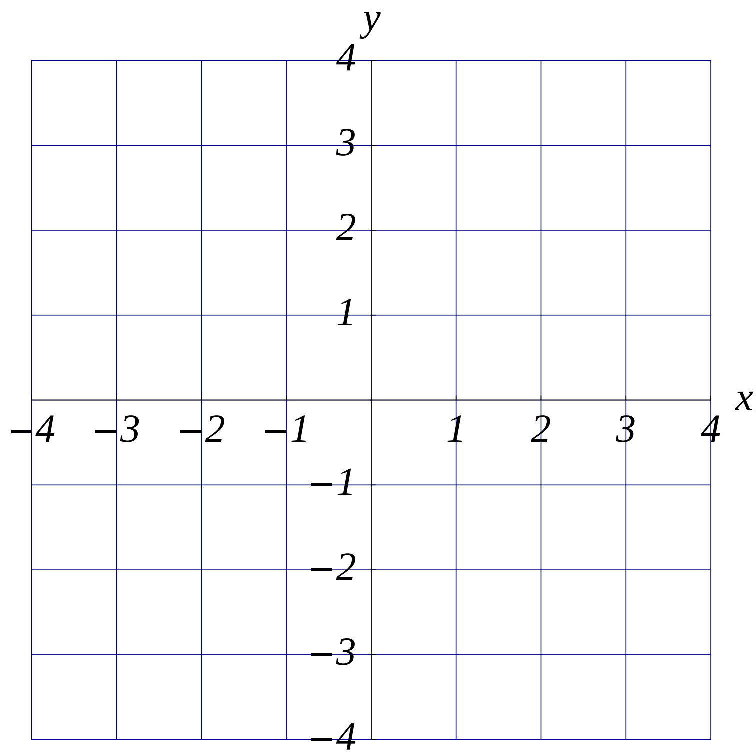
   (a) Write a system of differential equations that describes the situation when species $x(t)$ is a parasite and species $y(t)$ is a host and where the presence of the parasite is beneficial to the host.

   (b) Write a system of differential equations that describes the situation when species $x(t)$ is a parasite and species $y(t)$ is a host and where the presence of the parasite is harmful to the host.

2. (12 points) Consider the system of differential equations

$$\frac{dx}{dt} = -x - 2y,$$
$$\frac{dy}{dt} = 4x + 3y.$$

(a) Draw and label the x and y-nullclines on the graph below. Be sure to indicate the direction of the solution on the nullclines.



(b) Label the regions where $dx/dt > 0$ and where $dx/dt < 0$ on the graph. Do the same for dy/dt.

(c) Decide if $(0, 0)$ is a stable equilibrium point. Justify your answer.

(d) For the initial condition $x(0) = 0$ and $y(0) = -1$, sketch the trajectory in the phase plane on the graph on the previous page.

3. (20 points) Consider the system I

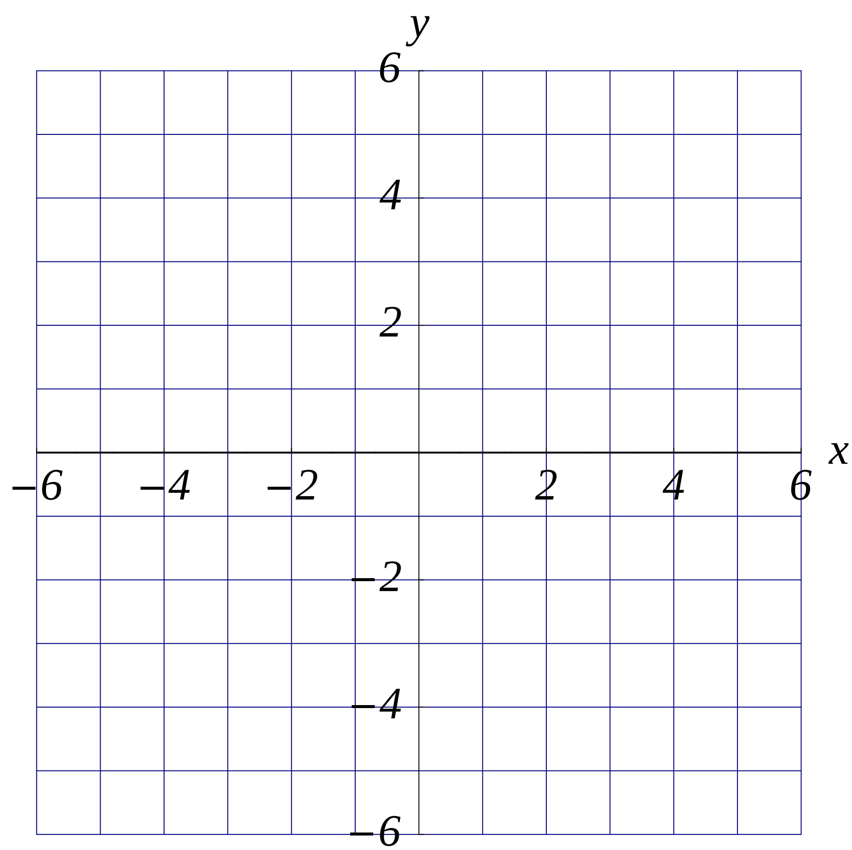
$$\begin{aligned} \frac{dx}{dt} &= 6x - 2x^2 - 3xy \\ \frac{dy}{dt} &= y - xy - y^2, \end{aligned}$$

(a) Find the x and y-nullclines of the system.

(b) Find all of the equilibrium solutions.

(c) Using linearization, determine the nature of each equilibrium solution.

(d) Sketch and *label* the nullclines on the graph below. Be sure to indicate the direction of the solution on the nullclines. Sketch the trajectory in xy-plane that begins at $(2, 2)$.



4. (8 points) Match equations and slope fields

(a) $\dfrac{dy}{dt} = 1 + y^2$

(b) $\dfrac{dy}{dt} = y^2 - t^2$

(c) $\dfrac{dy}{dt} = ty$

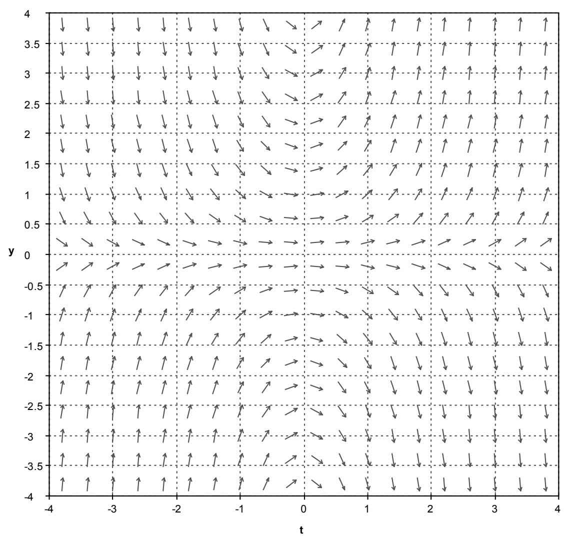
(d) $\dfrac{dy}{dt} = 1 - y$

(e) $\dfrac{dy}{dt} = y(1 - y) - 2$

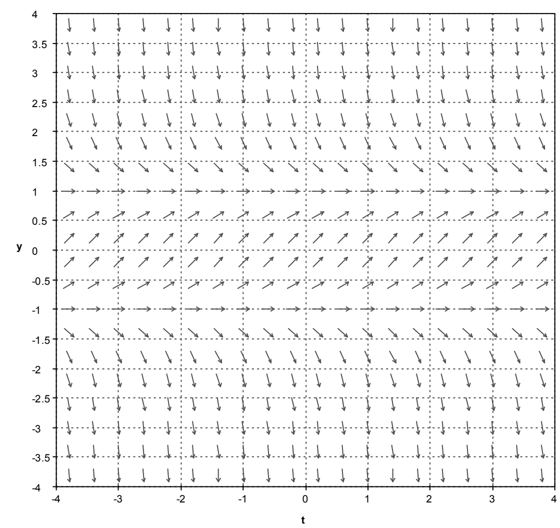
(f) $\dfrac{dy}{dt} = (y - t)^2$

(g) $\dfrac{dy}{dt} = 1 - y^2$

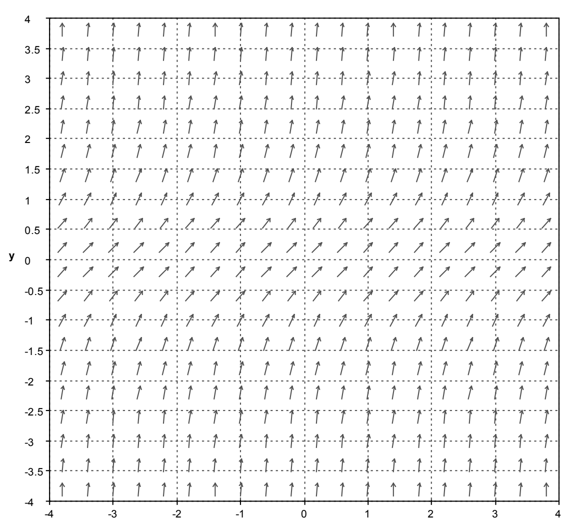
(h) $\dfrac{dy}{dt} = y^2 - t^2$



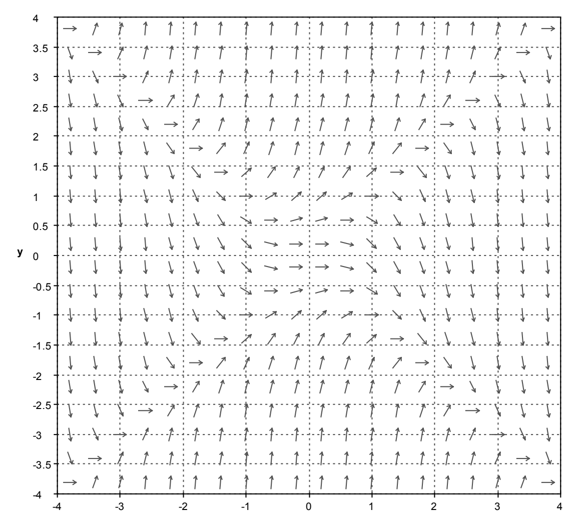
(i)



(ii)



(iii)



(iv)

5. (12 points) Compute each of the following partial derivatives:

(a) $\dfrac{\partial}{\partial x}(x^2 + \sin y + xe^y)$

(b) $\dfrac{\partial^2}{\partial x \partial y}(x^2 + \sin y + xe^y)$

(c) $\dfrac{\partial^2}{\partial x^2}(x^2 + \sin y + xe^y)$

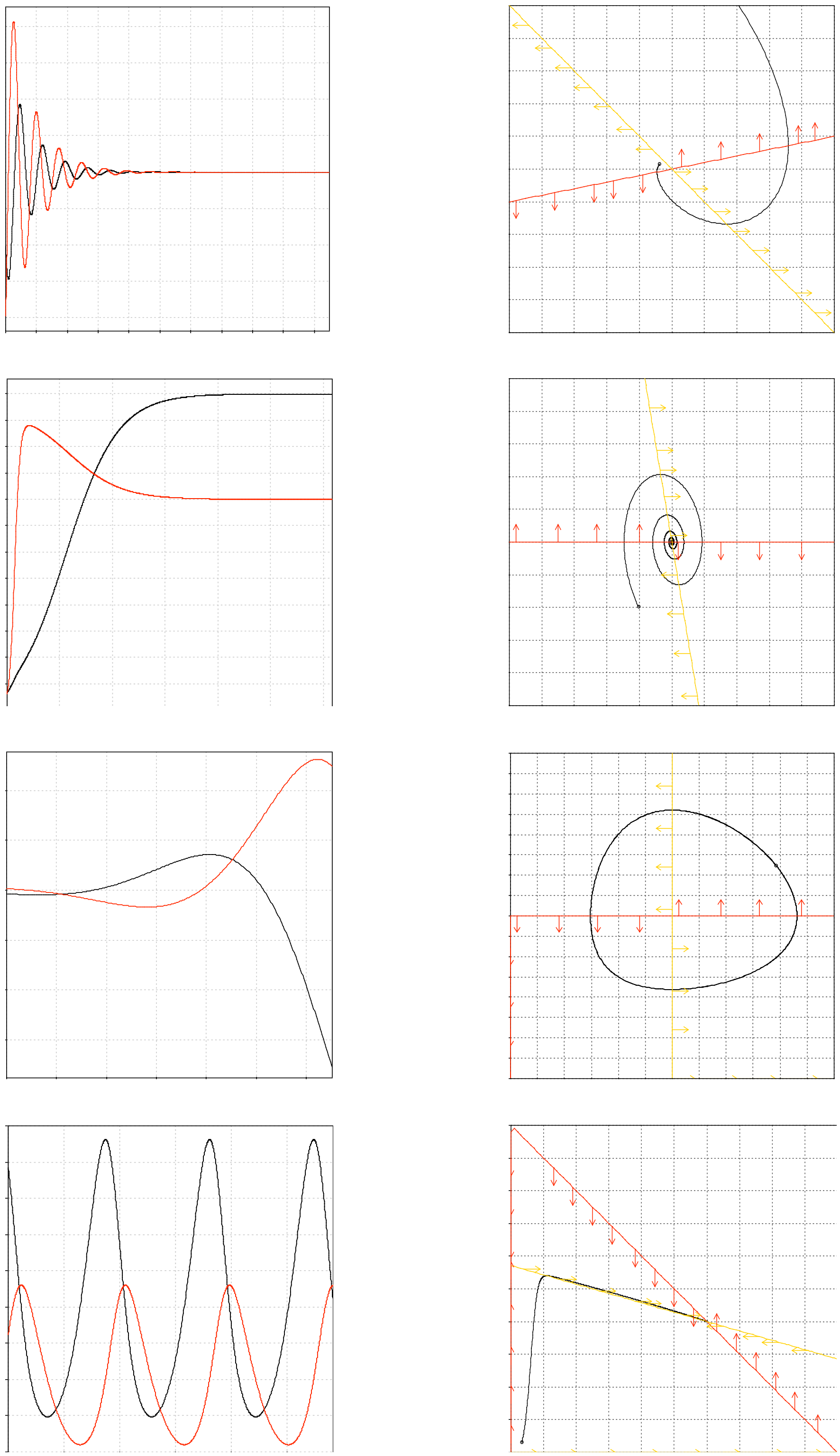
(d) $\dfrac{\partial^2}{\partial y^2}(x^2 + \sin y + xe^y)$

6. (8 points) Compute each of the following definite integrals:

(a) $\int_0^2 \int_0^1 4x^3 + 6xy^2 \, dxdy$

(b) $\int_0^1 \int_0^{\pi/2} e^y + \sin x \, dxdy$

7. (8 points) In the following images, match the $t \to (t, x(t))$ and $t \to (t, y(t))$ plots in the first column with the corresponding phase plane plots, $t \to (x(t), y(t))$ in the second column.



8. (12 points)

(a) Sketch a graph of $f(y) = (y + 1)(y - 9)^2$

(b) Use the graph in part (a) to develop a phase line for the autonomous differential equation

$$y' = (y + 1)(y - 9)^2.$$

(c) Classify each equilibrium solution of the differential equation $y' = (y+1)(y-9)^2$ as a source, a sink, or a node.

(d) Sketch the equilibrium solutions of $y' = (y+1)(y-9)^2$ in the ty-plane. These equilibrium solutions divide the ty-plane into regions. Sketch at least one solution trajectory in each of the regions.

9. (10 points) Suppose that you wish to model a population with a differential equation of the form $dP/dt = f(P)$, where $P(t)$ is the population at time t. Experiments have been performed on the population that give the following information:

- The population at $P = 0$ remains constant.
- A population close to 0 will decrease.
- A population of $P = 20$ does not change.
- A population of $20 < P < 100$ increases.
- A population of $P > 100$ will decrease.

(a) Sketch the simplest possible phase line that agrees with the experimental information above.

(b) Write a formula for $f(P)$ that agrees with the phase line in part (a).