

Math 19

Midterm II Review

T. Judson
December 12, 2005

Resources for Review

- * **Midterm II Review Guide**

<http://www.courses.fas.harvard.edu/math19/exams/review2.pdf>

- * **Exams and solutions from previous**

years <http://www.courses.fas.harvard.edu/~math19/prevexams/>

Exam Particulars

- * Thursday, December 16 at 7-9 PM in Science Center A
- * You may use a calculator on this exam, and you are permitted one 4 by 6 inch note card
- * All out-of-sequence exams must be approved
- * No make-up exams

What to Expect

- * Eight questions (some with several parts)
- * The emphasis will be on material from Chapters 13-24
- * Refer to the Midterm II Review Guide for details

Poincaré-Bendixson

Consider the predator-prey system

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{6}(x-1) \left(5 - x - \frac{6y}{x} \right) \\ \frac{dy}{dt} &= \frac{y}{20} \left(1 - \frac{y}{x-1} \right).\end{aligned}$$

- (a) Find the nullclines of the system.
- (b) Find all equilibrium points of this system.
- (c) Find an repelling equilibrium point for the system.
- (d) Make a careful argument to show that $1 \leq x \leq 5$ and $0 \leq y \leq 6$ is a basin of attraction and use the Poincaré-Bendixson Theorem to show that the system has a periodic solution

Superposition

Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u, \quad (1)$$

where $0 \leq x \leq \pi$ and $t \geq 0$.

- (a) Show that $u_n(t, x) = e^{-nt} \sin(\sqrt{1+n}x)$ is also a solution to equation (1) for $n = 1, 2, 3, \dots$
- (b) Use the Principle of Superposition and the functions $u_n(x)$ to construct a solution to equation (1) that satisfies the boundary and initial conditions

$$u(0, x) = 5 \sin 2x - 2 \sin 3x + \sin 4x,$$
$$u(t, 0) = u(t, \pi) = 0.$$

Separation of Variables

Consider the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 10u.$$

- (a) Use the separation of variables technique to find nonzero solutions to this equation subject to the conditions for all $t \geq 0$ and for $0 \leq x \leq L$ such that

$$\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, L) = 0.$$

Stability

Consider the equilibrium solution $u_e = -7$ to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(u + 7)$$

satisfying the boundary conditions

$$\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, L) = 0.$$

Determine the stability of this solution.

The Maximum Principle

Given the equation

$$\lambda g = \frac{d^2 g}{dx^2} - \frac{1}{1 + e^x} g,$$

use the Maximum Principle to determine if there is a positive number λ and a solution g defined on the interval $0 \leq x \leq L$ that vanishes at $x = 0$ and $x = L$ and is not identically zero. In particular, write *no* and justify your answer using the Maximum Principle or explain why the Maximum Principle cannot be applied.

Fast and Slow Systems

(a) Consider an equation, $x(t)$, as a function of time, of the form

$$\frac{dx}{dt} = x^4 - 8x^2 + c,$$

where c is a constant. Find all values of c where the number of equilibrium solutions changes.

(b) Given the system

$$\begin{aligned}x' &= 0.01x - 0.003xy \\y' &= y - 0.014xy,\end{aligned}$$

decide which equation represents the slow subsystem and which equation represents the fast subsystem. You may assume that $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$.

Advection

Find a function $u(t, x)$ that solve the advection equation

$$\frac{\partial u}{\partial t} = -2 \frac{\partial u}{\partial x} + 5u$$

and satisfies the initial condition

$$u(0, x) = e^{-x^2/2}.$$